

Supersymmetry Analogues of the Classical Theorem on Harmonic Polynomials ¹

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Abstract

Classical harmonic analysis says that the spaces of homogeneous harmonic polynomials (solutions of Laplace equation) are irreducible modules of the corresponding orthogonal Lie group (algebra) and the whole polynomial algebra is a free module over the invariant polynomials generated by harmonic polynomials. In this paper, we first establish two-parameter \mathbb{Z}^2 -graded supersymmetric oscillator generalizations of the above theorem for the Lie superalgebra $gl(n|m)$. Then we extend the result to two-parameter \mathbb{Z} -graded supersymmetric oscillator generalizations of the above theorem for the Lie superalgebras $osp(2n|2m)$ and $osp(2n+1|2m)$.

1 Introduction

Harmonic polynomials are important objects in analysis, differential geometry and physics. A fundamental theorem in classical harmonic analysis says that the spaces of homogeneous harmonic polynomials (solutions of Laplace equation) are irreducible modules of the corresponding orthogonal Lie group (algebra) and the whole polynomial algebra is a free module over the invariant polynomials generated by harmonic polynomials. Bases of these irreducible modules can be obtained easily (e.g., cf. [X1]). The algebraic beauty of the above theorem is that Laplace operator characterizes the irreducible submodules of the polynomial algebra and the corresponding quadratic invariant gives a decomposition of the polynomial algebra into a direct sum of irreducible submodules.

Cao [C] proved that the subspaces of homogeneous polynomial vector solutions of the n -dimensional Navier equations in elasticity are exactly direct sums of three explicitly given irreducible submodules when $n \neq 4$ and direct sums of four explicitly given irreducible submodules if $n = 4$ of the corresponding orthogonal Lie group (algebra), and the whole polynomial vector space is also a free module over the invariant polynomials generated these solutions. This is essentially a vector-function generalization of the classical theorem on harmonic polynomials.

In [X2], the second author proved that the space of homogeneous polynomial solutions with degree m for the dual cubic Dickson invariant differential operator is exactly a direct sum of $\llbracket m/2 \rrbracket + 1$ explicitly determined irreducible E_6 -submodules and the whole polynomial algebra is a free module over the polynomial algebra in the Dickson invariant

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generated by these solutions. This gave a cubic E_6 -generalization of the classical theorem on harmonic polynomials.

Lie algebras (Lie groups) serve as the symmetries in quantum physics (e.g., cf. [FC, L, LF, G]). Their various representations provide distinct concrete practical physical models. Many important physical phenomena have been interpreted as the consequences of symmetry breakings (e.g., cf. [LF]). Harmonic oscillators are basic objects in quantum mechanics (e.g., cf. [FC, G]). Oscillator representations of finite-dimensional simple Lie algebras are the most fundamental ones in quantum physics (e.g., cf. [DES, FSS]). Howe [Ho] obtained a \mathbb{Z} -graded multiplicity-free oscillator representation for $sl(n, \mathbb{C})$. In [X1], the second author found the methods of solving flag partial differential equations for polynomial solutions. Moreover, we [LX] used a result in [X1] to prove two-parameter \mathbb{Z}^2 -graded oscillator generalizations of the classical theorem on harmonic polynomials for the Lie algebra $sl(n, \mathbb{C})$ and two-parameter \mathbb{Z} -graded oscillator generalizations of the theorem for the Lie algebra $o(n, \mathbb{C})$.

The aim of this work is to establish two-parameter \mathbb{Z}^2 -graded supersymmetric oscillator generalizations of the classical theorem on harmonic polynomials for the Lie superalgebra $gl(n|m)$ and two-parameter \mathbb{Z} -graded supersymmetric oscillator generalizations of the theorem for the Lie superalgebras $osp(2n|2m)$ and $osp(2n+1|2m)$. Below we give a technical introduction.

Suppose that $n \geq 3$ is an integer. Denote by $E_{r,s}$ the square matrix with 1 as its (r, s) -entry and 0 as the others. The compact orthogonal Lie algebra

$$o(n, \mathbb{R}) = \sum_{1 \leq r < s \leq n} \mathbb{R}(E_{r,s} - E_{s,r}). \quad (1.1)$$

It acts on the polynomial algebra $\mathcal{A} = \mathbb{R}[x_1, \dots, x_n]$ by $(E_{r,s} - E_{s,r})|_{\mathcal{A}} = x_r \partial_{x_s} - x_s \partial_{x_r}$. Recall the Laplace operator

$$\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \dots + \partial_{x_n}^2. \quad (1.2)$$

Moreover, we have the fundamental invariant

$$\eta = x_1^2 + x_2^2 + \dots + x_n^2. \quad (1.3)$$

Denote by \mathcal{A}_k the subspace of homogeneous polynomials in \mathcal{A} with degree k . Classical theorem on harmonic polynomials says that the subspace of harmonic polynomials

$$\mathcal{H}_k = \{f \in \mathcal{A}_k \mid \Delta(f) = 0\} \quad (1.4)$$

forms an irreducible $o(n, \mathbb{R})$ -module and $\mathcal{A} = \bigoplus_{i,k=0}^{\infty} \eta^i \mathcal{H}_k$ is a direct sum of irreducible $o(n, \mathbb{R})$ -submodules. The beauty of the above theorem is that the invariant differential operator Δ characterizes the irreducible submodules and its dual operator η gives the complete reducibility.

Fix two positive integers m and n . Set

$$gl(n|m)_0 = \sum_{i,j=1}^n \mathbb{C}E_{i,j} + \sum_{r,s=1}^m \mathbb{C}E_{n+r,n+s} \quad (1.5)$$

and

$$gl(n|m)_1 = \sum_{i=1}^n \sum_{r=1}^m (\mathbb{C}E_{i,n+r} + \mathbb{C}E_{n+r,i}). \quad (1.6)$$

The Lie superalgebra $gl(n|m) = gl(n|m)_0 + gl(n|m)_1$ with the algebraic operation $[\cdot, \cdot]$ defined by

$$[A, B] = AB - (-1)^{i_1 i_2} BA \quad \text{for } A \in gl(n|m)_{i_1}, B \in gl(n|m)_{i_2}. \quad (1.7)$$

For convenience, we use the notion $\overline{i, i+1} = \{i, i+1, i+2, \dots, i+j\}$ for integers i and j with $i \leq j$. Let \mathcal{A} be the polynomial algebra in bosonic variables $\{x_i \mid i \in \overline{1, 2n}\}$ and fermionic variables $\{\theta_j \mid j \in \overline{1, 2m}\}$, i.e.,

$$x_r x_s = x_s x_r, \quad \theta_p \theta_q = -\theta_q \theta_p, \quad x_r \theta_p = \theta_p x_r \quad (1.8)$$

for $r, s \in \overline{1, 2n}$ and $p, q \in \overline{1, 2m}$. Set $\Theta = \sum_{p=1}^{2m} \mathbb{C} \theta_p$. Write

$$\mathcal{A}_{(0)} = \sum_{q=0}^m \mathbb{C}[x_1, \dots, x_{2n}] \Theta^{2q}, \quad \mathcal{A}_{(1)} = \sum_{q=1}^{m-1} \mathbb{C}[x_1, \dots, x_{2n}] \Theta^{2q+1}. \quad (1.9)$$

Then $\mathcal{A} = \mathcal{A}_{(0)} \oplus \mathcal{A}_{(1)}$ is a \mathbb{Z}_2 -graded algebra.

For $r \in \overline{1, n}$, the usual differential operator ∂_{x_r} acts on \mathcal{A} as a derivation such that $\partial_{x_r}(x_s) = \delta_{r,s}$ and $\partial_{x_r}(\theta_p) = 0$ for $s \in \overline{1, 2n}$ and $p \in \overline{1, 2m}$. Moreover, for $p \in \overline{1, 2m}$, we define ∂_{θ_p} as a linear operator on \mathcal{A} with $\partial_{\theta_p}(x_r) = 0$ and $\partial_{\theta_p}(\theta_q) = \delta_{p,q}$ for $r \in \overline{1, n}$ and $q \in \overline{1, 2m}$, such that

$$\partial_{\theta_p}(fg) = \partial_{\theta_p}(f)g + (-1)^t f \partial_{\theta_p}(g) \quad \text{for } f \in \mathcal{A}_{(t)}, g \in \mathcal{A}. \quad (1.10)$$

For later notational convenience, we redenote

$$y_i = x_{n+i}, \quad \vartheta_j = \theta_{m+j} \quad \text{for } i \in \overline{1, n}, j \in \overline{1, m}. \quad (1.11)$$

Define a representation of $gl(n|m)$ on \mathcal{A} determined by

$$E_{i,j}|_{\mathcal{A}} = x_i \partial_{x_j} - y_j \partial_{y_i}, \quad E_{i,n+r}|_{\mathcal{A}} = x_i \partial_{\theta_r} - \vartheta_r \partial_{y_i}, \quad (1.12)$$

$$E_{n+r,i}|_{\mathcal{A}} = \theta_r \partial_{x_i} + y_i \partial_{\vartheta_r}, \quad E_{n+r,n+s}|_{\mathcal{A}} = \theta_r \partial_{\theta_s} - \vartheta_s \partial_{\vartheta_r} \quad (1.13)$$

for $i, j \in \overline{1, n}$ and $r, s \in \overline{1, m}$.

Write $\Theta_1 = \sum_{r=1}^m \mathbb{C} \theta_r$ and $\Theta_2 = \sum_{s=1}^m \mathbb{C} \vartheta_s$. Denote by \mathbb{N} the set of nonnegative integers. For $\ell_1, \ell_2 \in \mathbb{N}$, we denote

$$\mathcal{A}_{\ell_1, \ell_2} = \text{Span}\{x^\alpha y^\alpha \Theta_1^{\ell'_1} \Theta_2^{\ell'_2} \mid \alpha, \beta \in \mathbb{N}^n; \ell'_1, \ell'_2 \in \mathbb{N}; |\alpha| + \ell'_1 = \ell_1, |\beta| + \ell'_2 = \ell_2\}, \quad (1.14)$$

where $|\gamma| = \sum_{i=1}^n \gamma_i$ for $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}^n$. Let

$$\Delta = \sum_{i=1}^n \partial_{x_i} \partial_{y_i} + \sum_{r=1}^m \partial_{\theta_r} \partial_{\vartheta_r}, \quad \eta = \sum_{i=1}^n x_i y_i + \sum_{r=1}^m \theta_r \vartheta_r. \quad (1.15)$$

Moreover, we define

$$\mathcal{H}_{\ell_1, \ell_2} = \{f \in \mathcal{A}_{\ell_1, \ell_2} \mid \Delta(f) = 0\}. \quad (1.16)$$

Denote $k_{\ell, \ell'} = \min\{\ell, \ell'\}$. The following is our first main result.

Theorem 1. *Let $\ell, \ell' \in \mathbb{N}$. The space $\mathcal{H}_{\ell, \ell'}$ is an irreducible $gl(n|m)$ -module if and only if $\ell > m + 1 - n$ or $\ell' > m + 1 - n$ or $\ell + \ell' \leq m + 1 - n$. When $|\ell - \ell'| > m + 1 - n$*

or $\ell + \ell' \leq m + 1 - n$, $\mathcal{A}_{\ell, \ell'} = \bigoplus_{i=0}^{k_{\ell, \ell'}} \eta^i \mathcal{H}_{\ell-i, \ell'-i}$ is a decomposition of irreducible $gl(n|m)$ -submodules.

We remark that if $\ell, \ell' \leq m+1-n$ and $\ell + \ell' > m+1-n$, then $\mathcal{H}_{\ell, \ell'}$ is an indecomposable $gl(n|m)$ -module. In fact, $\mathcal{H}_{\ell, \ell'} \cap \eta \mathcal{A}_{\ell-1, \ell'-1} \neq \{0\}$. This also shows that $\mathcal{A}_{\ell, \ell'}$ is not completely reducible when $|\ell - \ell'| \leq m + 1 - n$ and $\ell + \ell' > m + 1 - n$.

Fix $1 < n_1 + 1 < n_2 \leq n$. Note

$$[\partial_{x_r}, x_r] = 1 = [-x_r, \partial_{x_r}], \quad [\partial_{y_s}, y_s] = 1 = [-y_s, \partial_{y_s}]. \quad (1.17)$$

Changing operators $\partial_{x_r} \mapsto -x_r$, $x_r \mapsto \partial_{x_r}$ for $r \in \overline{1, n_1}$ and $\partial_{y_s} \mapsto -y_s$, $y_s \mapsto \partial_{y_s}$ for $s \in \overline{n_2 + 1, n}$ in (1.12) and (1.13), we get a new representation of $gl(n|m)$ on \mathcal{A} determined by

$$E_{i,j}|_{\mathcal{A}} = E_{i,j}^x - E_{j,i}^y, \quad E_{n+r, n+s}|_{\mathcal{A}} = \theta_r \partial_{\theta_s} - \vartheta_s \partial_{\theta_r} \quad (1.18)$$

with

$$E_{i,j}^x|_{\mathcal{A}} = \begin{cases} -x_j \partial_{x_i} - \delta_{i,j} & \text{if } i, j \in \overline{1, n_1}; \\ \partial_{x_i} \partial_{x_j} & \text{if } i \in \overline{1, n_1}, j \in \overline{n_1 + 1, n}; \\ -x_i x_j & \text{if } i \in \overline{n_1 + 1, n}, j \in \overline{1, n_1}; \\ x_i \partial_{x_j} & \text{if } i, j \in \overline{n_1 + 1, n} \end{cases} \quad (1.19)$$

and

$$E_{i,j}^y|_{\mathcal{A}} = \begin{cases} y_i \partial_{y_j} & \text{if } i, j \in \overline{1, n_2}; \\ -y_i y_j & \text{if } i \in \overline{1, n_2}, j \in \overline{n_2 + 1, n}; \\ \partial_{y_i} \partial_{y_j} & \text{if } i \in \overline{n_2 + 1, n}, j \in \overline{1, n_2}; \\ -y_j \partial_{y_i} - \delta_{i,j} & \text{if } i, j \in \overline{n_2 + 1, n}; \end{cases} \quad (1.20)$$

$$E_{i, n+r}|_{\mathcal{A}} = \begin{cases} \partial_{x_i} \partial_{\theta_r} - \vartheta_r \partial_{y_i} & \text{if } i \in \overline{1, n_1}; \\ x_i \partial_{\theta_r} - \vartheta_r \partial_{y_i} & \text{if } i \in \overline{n_1 + 1, n_2}; \\ x_i \partial_{\theta_r} + y_i \vartheta_r & \text{if } i \in \overline{n_2 + 1, n}; \end{cases} \quad (1.21)$$

$$E_{n+r, i}|_{\mathcal{A}} = \begin{cases} -x_i \theta_r + y_i \partial_{\theta_r} & \text{if } i \in \overline{1, n_1}; \\ \theta_r \partial_{x_i} + y_i \partial_{\theta_r} & \text{if } i \in \overline{n_1 + 1, n_2}; \\ \theta_r \partial_{x_i} + \partial_{y_i} \partial_{\theta_r} & \text{if } i \in \overline{n_2 + 1, n} \end{cases} \quad (1.22)$$

for $i, j \in \overline{1, n}$ and $r, s \in \overline{1, m}$.

The related Laplace operator becomes

$$\Delta = - \sum_{i=1}^{n_1} x_i \partial_{y_i} + \sum_{r=n_1+1}^{n_2} \partial_{x_r} \partial_{y_r} - \sum_{s=n_2+1}^n y_s \partial_{x_s} + \sum_{r=1}^m \partial_{\theta_r} \partial_{\vartheta_r} \quad (1.23)$$

and its dual

$$\eta = \sum_{i=1}^{n_1} y_i \partial_{x_i} + \sum_{r=n_1+1}^{n_2} x_r y_r + \sum_{s=n_2+1}^n x_s \partial_{y_s} + \sum_{r=1}^m \theta_r \vartheta_r. \quad (1.24)$$

Denote

$$\begin{aligned} \mathcal{A}_{\langle \ell_1, \ell_2 \rangle} &= \text{Span}\{x^\alpha y^\beta \Theta_1^{\ell'_1} \Theta_2^{\ell'_2} \mid \alpha, \beta \in \mathbb{N}^n; \ell'_1, \ell'_2 \in \mathbb{N}; \\ &\quad \sum_{r=n_1+1}^n \alpha_r - \sum_{i=1}^{n_1} \alpha_i + \ell'_1 = \ell_1; \sum_{i=1}^{n_2} \beta_i - \sum_{r=n_2+1}^n \beta_r + \ell'_2 = \ell_2\} \end{aligned} \quad (1.25)$$

for $\ell_1, \ell_2 \in \mathbb{Z}$. Again we set $\mathcal{H}_{\langle \ell_1, \ell_2 \rangle} = \{f \in \mathcal{A}_{\langle \ell_1, \ell_2 \rangle} \mid \Delta(f) = 0\}$. The following is our second main result.

Theorem 2. Let $\ell, \ell' \in \mathbb{Z}$ such that $\ell' \geq 0$ if $n_2 = n$. The $gl(n|m)$ -module $\mathcal{H}_{\langle \ell, \ell' \rangle}$ is irreducible if and only if $\ell + \ell' \leq n_1 + m + 1 - n_2$ or $\ell \notin \overline{n_1 + 1 - n, n_1 + m + 1 - n}$ and $n_2 = n$. When $\ell + \ell' \leq n_1 + m + 1 - n_2$, $\mathcal{A}_{\langle \ell, \ell' \rangle} = \bigoplus_{i=0}^{\infty} \eta^i(\mathcal{H}_{\langle \ell-i, \ell'-i \rangle})$ is the decomposition of irreducible $gl(n|m)$ -submodules if $n_2 < n$, and $\mathcal{A}_{\langle \ell, \ell' \rangle} = \bigoplus_{i=0}^{\ell'} \eta^i(\mathcal{H}_{\langle \ell-i, \ell'-i \rangle})$ is the decomposition of irreducible $gl(n|m)$ -submodules if $n_2 = n$.

If $\ell + \ell' > n_1 + m + 1 - n_2$ and $\ell \in \overline{n_1 + 1 - n, n_1 + m + 1 - n}$ when $n_2 = n$, the $gl(n|m)$ -module $\mathcal{H}_{\langle \ell, \ell' \rangle}$ is indecomposable. When $n_2 < n$ and $\ell + \ell' > n_1 + m + 1 - n_2$, $\mathcal{A}_{\langle \ell, \ell' \rangle}$ is not completely reducible.

We use (1.14) and (1.15) to define

$$\mathcal{A}_k = \bigoplus_{\ell=0}^k \mathcal{A}_{\ell, k-\ell}, \quad \mathcal{H}_k = \{f \in \mathcal{A}_k \mid \Delta(f) = 0\} \quad (1.26)$$

for $k \in \mathbb{N}$. Moreover, we use (1.23) and (1.25) to define

$$\mathcal{A}_{\langle k \rangle} = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{A}_{\langle \ell, k-\ell \rangle}, \quad \mathcal{H}_{\langle k \rangle} = \{f \in \mathcal{A}_{\langle k \rangle} \mid \Delta(f) = 0\} \quad (1.27)$$

for $k \in \mathbb{Z}$. The above representations of $gl(n|m)$ can be uniquely extended to the representations of the Lie superalgebra $osp(2n|2m)$.

Theorem 3. Suppose $n > 1$. For $k \in \mathbb{N}$, \mathcal{H}_k is an irreducible $osp(2n|2m)$ -module if and only if $k \leq m + 1 - n$ or $k > 2(m + 1 - n)$. When $k \leq m + 1 - n$, $\mathcal{A}_k = \bigoplus_{i=0}^{\lfloor k/2 \rfloor} \eta^i \mathcal{H}_{k-2i}$ is a decomposition of irreducible $osp(2n|2m)$ -submodules.

Let $k \in \mathbb{Z}$. The $osp(2n|2m)$ -module $\mathcal{H}_{\langle k \rangle}$ is irreducible if and only if $k \leq n_1 + m + 1 - n_2$. When $k \leq n_1 + m + 1 - n_2$, $\mathcal{A}_{\langle k \rangle} = \bigoplus_{i=0}^{\infty} \eta^i(\mathcal{H}_{\langle k-2i \rangle})$ is the decomposition of irreducible $osp(2n|2m)$ -submodules.

Let x_0 be a bosonic (commuting) variable. Set

$$\mathcal{B} = \mathcal{A}[x_0] = \bigoplus_{k=0}^{\infty} \mathcal{B}_k, \quad \mathcal{B}_k = \sum_{i=0}^k \mathcal{A}_{k-i} x_0^i. \quad (1.28)$$

Moreover, the corresponding supersymmetric Laplace operator and invariant become

$$\Delta' = \partial_{x_0}^2 + 2 \sum_{i=1}^n \partial_{x_i} \partial_{y_i} + 2 \sum_{r=1}^m \partial_{\theta_r} \partial_{\vartheta_r}, \quad \eta' = x_0^2 + 2 \sum_{i=1}^n x_i y_i + 2 \sum_{r=1}^m \theta_r \vartheta_r. \quad (1.29)$$

Then the representation of $gl(n|m)$ given in (1.12) and (1.13) can be uniquely extended to a representation of $osp(2n+1|2m)$ on \mathcal{B} such that Δ' is an $osp(2n+1|2m)$ -invariant operator and η' is an $osp(2n+1|2m)$ -invariant. Denote

$$\mathcal{H}'_k = \{f \in \mathcal{B}_k \mid \Delta'(f) = 0\} \quad (1.30)$$

for $k \in \mathbb{N}$.

Similarly, the representation of $gl(n|m)$ given in (1.18)-(1.22) can be uniquely extended to a representation of $osp(2n+1|2m)$ on \mathcal{B} such that the operators

$$\Delta' = \partial_{x_0}^2 - 2 \left(\sum_{i=1}^{n_1} x_i \partial_{y_i} + \sum_{r=n_1+1}^{n_2} \partial_{x_r} \partial_{y_r} - \sum_{s=n_2+1}^n y_s \partial_{x_s} + \sum_{r=1}^m \partial_{\theta_r} \partial_{\vartheta_r} \right) \quad (1.31)$$

and

$$\eta' = x_0^2 + 2\left(\sum_{i=1}^{n_1} y_i \partial_{x_i} + \sum_{r=n_1+1}^{n_2} x_r y_r + \sum_{s=n_2+1}^n x_s \partial_{y_s} + \sum_{r=1}^m \theta_r \vartheta_r\right) \quad (1.32)$$

are $osp(2n+1|2m)$ -invariant operators. Set

$$\mathcal{B}_{\langle k \rangle} = \sum_{i=0}^{\infty} \mathcal{A}_{\langle k-i \rangle} x_0^i, \quad \mathcal{H}'_{\langle k \rangle} = \{f \in \mathcal{B}_{\langle k \rangle} \mid \Delta'(f) = 0\}. \quad (1.33)$$

Theorem 4. *For any $k \in \mathbb{N}$, \mathcal{H}'_k is an irreducible $osp(2n+1|2m)$ -module. Moreover, $\mathcal{B} = \bigoplus_{\ell,k=0}^{\infty} (\eta')^{\ell} \mathcal{H}_k$ is a direct sum of irreducible $osp(2n+1|2m)$ -submodules.*

For any $k \in \mathbb{Z}$, $\mathcal{H}'_{\langle k \rangle}$ is an irreducible $osp(2n+1|2m)$ -module. Moreover, $\mathcal{B} = \bigoplus_{\ell,k=0}^{\infty} (\eta')^{\ell} \mathcal{H}_{\langle k \rangle}$ is a direct sum of irreducible $osp(2n+1|2m)$ -submodules.

The first conclusion in Theorem 3 with $n > m+1$ and the first conclusion in Theorem 4 with $n > m$ were obtained by Zhang [Z].

In Section 2, we give the proof of Theorem 1. We prove Theorem 2 in Section 3. Section 4 is devoted to the proof of Theorem 3. We show Theorem 4 in Section 5.

2 Proof of Theorem 1

In this section, we want to prove Theorem 1. Recall the settings in (1.5)-(1.16).

Set

$$\bar{\mathcal{A}} = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n], \quad \check{\mathcal{A}} = \sum_{i=0}^{2m} \Theta^{2m}. \quad (2.1)$$

Then $\bar{\mathcal{A}}$ and $\check{\mathcal{A}}$ are subalgebras of \mathcal{A} , and $\mathcal{A} = \bar{\mathcal{A}}\check{\mathcal{A}}$. It is straightforward to verify

$$E_{i,j} \Delta = \Delta E_{i,j}, \quad E_{i,j} \eta = \eta E_{i,j} \quad \text{for } i, j \in \overline{1, m+n} \quad (2.2)$$

by (1.12), (1.13) and (1.15). Indeed, η is an invariant, that is, $E_{i,j}(\eta) = 0$ for any $i, j \in \overline{1, m+n}$. Write

$$\bar{\mathcal{G}} = \sum_{i,j=1}^n \mathbb{C} E_{i,j}, \quad \check{\mathcal{G}} = \sum_{r,s=1}^m \mathbb{C} E_{n+r,n+s}. \quad (2.3)$$

Then they are Lie subalgebras of $gl(n|m)$. Let

$$\bar{H} = \sum_{i=1}^n \mathbb{C} E_{i,i}, \quad \check{H} = \sum_{r=1}^m \mathbb{C} E_{n+r,n+s}, \quad (2.4)$$

$$\bar{\mathcal{G}}_+ = \sum_{1 \leq i < j \leq n} \mathbb{C} E_{i,j}, \quad \check{\mathcal{G}}_+ = \sum_{1 \leq r < s \leq m} \mathbb{C} E_{n+r,n+s}. \quad (2.5)$$

We take \bar{H} as a Cartan subalgebra of $\bar{\mathcal{G}}$ and $\bar{\mathcal{G}}_+$ as the subalgebra spanned by positive root vectors in $\bar{\mathcal{G}}$. Similarly, we take \check{H} as a Cartan subalgebra of $\check{\mathcal{G}}$ and $\check{\mathcal{G}}_+$ as the subalgebra spanned by positive root vectors in $\check{\mathcal{G}}$.

Let

$$\bar{\Delta} = \sum_{i=1}^n \partial_{x_i} \partial_{y_i}, \quad \bar{\eta} = \sum_{i=1}^n x_i y_i. \quad (2.6)$$

Recall

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad y^\beta = y_1^{\beta_1} \cdots y_n^{\beta_n} \quad \text{for } \alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n. \quad (2.7)$$

For $\ell_1, \ell_2 \in \mathbb{N}$, we denote

$$\bar{\mathcal{A}}_{\ell_1, \ell_2} = \text{Span}\{x^\alpha y^\beta \mid \alpha, \beta \in \mathbb{N}^n; \sum_{i=1}^n \alpha_i = \ell_1; \sum_{i=1}^n \beta_i = \ell_2\}. \quad (2.8)$$

Define

$$\bar{\mathcal{H}}_{\ell_1, \ell_2} = \{f \in \bar{\mathcal{A}}_{\ell_1, \ell_2} \mid \bar{\Delta}(f) = 0\}. \quad (2.9)$$

For $i \in \overline{1, n}$, we define $\varepsilon_i \in \bar{H}^*$ by

$$\bar{\varepsilon}_i(E_{j,j}) = \delta_{i,j} \quad \text{for } j \in \overline{1, n}. \quad (2.10)$$

We have the following lemma (e.g., cf. [X1]):

Lemma 2.1. *Suppose $n > 1$. For any ℓ_1, ℓ_2 , $\bar{\mathcal{H}}_{\ell_1, \ell_2}$ is a finite-dimensional irreducible $\bar{\mathcal{G}}$ -module with highest-weight vector $x_1^{\ell_1} y_n^{\ell_2}$ of weight $\ell_1 \varepsilon_1 + \ell_2 \varepsilon_n$. Moreover,*

$$\bar{\mathcal{A}} = \bigoplus_{\ell_1, \ell_2, \ell_3=0}^{\infty} \bar{\eta}^{\ell_1} \bar{\mathcal{H}}_{\ell_2, \ell_3} \quad (2.11)$$

is a decomposition of irreducible $\bar{\mathcal{G}}$ -submodules.

When $n = 1$, we have

$$\bar{\mathcal{H}}_{\ell, 0} = \mathbb{C}x_1^\ell, \quad \bar{\mathcal{H}}_{0, \ell} = \mathbb{C}y_1^\ell \quad \text{for } \ell \in \mathbb{N}. \quad (2.12)$$

Moreover,

$$\bar{\mathcal{A}} = \bigoplus_{\ell_1, \ell_2=0}^{\infty} (\bar{\eta}^{\ell_1} \bar{\mathcal{H}}_{\ell_2, 0} \oplus \bar{\eta}^{\ell_1} \bar{\mathcal{H}}_{0, \ell_2+1}). \quad (2.13)$$

Denote

$$\Theta_1 = \sum_{i=1}^m \mathbb{C}\theta_i, \quad \Theta_2 = \sum_{i=1}^m \mathbb{C}\vartheta_i. \quad (2.14)$$

For $\ell_1, \ell_2 \in \overline{1, m}$, we define

$$\check{\mathcal{A}}_{\ell_1, \ell_2} = \Theta_1^{\ell_1} \Theta_2^{\ell_2}. \quad (2.15)$$

Then $\check{\mathcal{A}}_{\ell_1, \ell_2}$ is a finite-dimensional $\check{\mathcal{G}}$ -module and

$$\check{\mathcal{A}} = \bigoplus_{\ell_1, \ell_2=0}^m \check{\mathcal{A}}_{\ell_1, \ell_2}. \quad (2.16)$$

Moreover, we define an ordering:

$$\theta_1 \prec \theta_2 \prec \cdots \prec \theta_m \prec \vartheta_m \prec \vartheta_{m-1} \prec \cdots \prec \vartheta_1. \quad (2.17)$$

On the basis

$$\begin{aligned} & \{\theta_{i_1} \cdots \theta_{i_r} \vartheta_{j_1} \cdots \vartheta_{j_s} \mid r, s \in \overline{0, m}; \\ & 1 \leq i_1 < i_2 < \cdots < i_r \leq m; m \geq j_1 > j_2 > \cdots > j_s \geq 1\} \end{aligned} \quad (2.18)$$

of $\check{\mathcal{A}}$, we define the partial ordering “ \prec ” lexically.

Write

$$\check{\Delta} = \sum_{r=1}^m \partial_{\theta_r} \partial_{\vartheta_r}, \quad \check{\eta} = \sum_{r=1}^m \theta_r \vartheta_r. \quad (2.19)$$

For $r \in \overline{1, m}$, we define

$$\vec{\theta}_r = \theta_1 \cdots \theta_r, \quad \vec{\vartheta}_r = \vartheta_m \cdots \vartheta_r. \quad (2.20)$$

For convenience, we let

$$\vec{\theta}_0 = 1 = \vec{\vartheta}_{m+1}. \quad (2.21)$$

It can easily proved that a minimal term of any singular vector in $\check{\mathcal{A}}_{\ell_1, \ell_2}$ is of the form $\vec{\theta}_r \vec{\vartheta}_s$ for some $r \in \overline{0, m}$ and $s \in \overline{1, m+1}$ or

$$\vec{\theta}_r \theta_{r+1} \cdots \theta_{s_1} \vec{\vartheta}_{s_2} \vartheta_{s_1+1} \cdots \vartheta_{r+1} \quad (2.22)$$

for some $0 \leq r < s_1 < s_2 \leq m+1$. A $\check{\mathcal{G}}$ -singular vector v is a nonzero weight vector of $\check{\mathcal{G}}$ such that $\check{\mathcal{G}}_+(v) = 0$. We count singular vector up to a nonzero scalar multiple. By comparing minimal terms, we can prove that

$$\{\check{\eta}^\ell \vec{\theta}_r \vec{\vartheta}_s \mid 0 \leq r < s \leq m+1; \ell \in \overline{0, s-r-1}; r+\ell = \ell_1; \ell+m-s+1 = \ell_2\} \quad (2.23)$$

is the set of all $\check{\mathcal{G}}$ -singular vectors in $\check{\mathcal{A}}_{\ell_1, \ell_2}$. Let $V_{r,s}$ be the finite-dimensional irreducible $\check{\mathcal{G}}$ -submodule generated by $\vec{\theta}_r \vec{\vartheta}_s \in \check{\mathcal{A}}_{r, m+1-s}$. By Weyl's Theorem of complete reducibility (e.g., cf. [Hu]),

$$\check{\mathcal{A}} = \bigoplus_{0 \leq r < s \leq m+1} \bigoplus_{\ell=0}^{s-r-1} \check{\eta}^\ell V_{r,s} \quad (2.24)$$

is a direct sum of irreducible $\check{\mathcal{G}}$ -submodules.

Define

$$\check{\mathcal{H}} = \{f \in \check{\mathcal{A}} \mid \check{\Delta}(f) = 0\}. \quad (2.25)$$

Note

$$E_{n+r, n+s} \check{\Delta} = \check{\Delta} E_{n+r, n+s}, \quad E_{n+r, n+s} \check{\eta} = \check{\eta} E_{n+r, n+s} \quad \text{on } \mathcal{A} \quad (2.26)$$

for $r, s \in \overline{1, m}$. Moreover,

$$\check{\Delta} \check{\eta} = \check{\eta} \check{\Delta} - m + \sum_{r=1}^m (\theta_r \partial_{\theta_r} + \vartheta_r \partial_{\vartheta_r}) \quad (2.27)$$

by (2.19). Furthermore,

$$\check{\Delta}(\vec{\theta}_r \vec{\vartheta}_s) = 0 \quad \text{if } r < s. \quad (2.28)$$

Hence

$$V_{r,s} \subset \check{\mathcal{H}} \quad \text{for } 0 \leq r < s \leq m+1 \quad (2.29)$$

by (2.26). Suppose $0 \leq r+1 < s \leq m+1$ and $\ell \in \overline{1, s-r-1}$. For any $f \in V_{r,s}$, we have

$$\check{\Delta}(\check{\eta}^\ell f) = \left(\sum_{p=0}^{\ell-1} (2p+r+1-s) \right) \check{\eta}^{\ell-1} f = \ell(\ell+r-s) \check{\eta}^{\ell-1} f. \quad (2.30)$$

Therefore,

$$\check{\mathcal{H}} = \bigoplus_{0 \leq r < s \leq m+1} V_{r,s}. \quad (2.31)$$

In particular,

$$\check{\mathcal{H}}_{r,m+1-s} = \{f \in \check{\mathcal{A}}_{r,m+1-s} \mid \check{\Delta}(f) = 0\} = \check{\mathcal{A}}_{r,m+1-s} \cap \check{\mathcal{H}} = V_{r,s} \quad (2.32)$$

for $0 \leq r < s \leq m+1$ and

$$\check{\mathcal{A}}_{\ell_1, \ell_2} \cap \check{\mathcal{H}} = \{0\} \quad \text{if } \ell_1 + \ell_2 \geq m+1. \quad (2.33)$$

For $r \in \overline{1, m}$, we define $\varepsilon'_r \in \check{H}^*$ by

$$\varepsilon'_r(E_{n+s, n+s}) = \delta_{r,s} \quad \text{for } s \in \overline{1, m}. \quad (2.34)$$

Moreover, we treat $\varepsilon'_0 = \varepsilon'_{m+1} = 0$. Then we have:

Lemma 2.2. *For $0 \leq r < s \leq m+1$, $\check{\mathcal{H}}_{r,m+1-s}$ is a finite-dimensional irreducible $\check{\mathcal{G}}$ -module with the highest-weight vector $\vec{\theta}_r \vec{\vartheta}_s$ of weight $\sum_{p=0}^r \varepsilon'_p - \sum_{q=s}^{m+1} \varepsilon'_q$. Moreover,*

$$\check{\mathcal{A}} = \bigoplus_{0 \leq r < s \leq m+1} \bigoplus_{\ell=0}^{s-r-1} \check{\eta}^\ell \check{\mathcal{H}}_{r,m+1-s}. \quad (2.35)$$

Recall

$$\bar{\Delta} \bar{\eta} = \bar{\eta} \bar{\Delta} + n + \sum_{i=1}^n (x_i \partial_{x_i} + y_i \partial_{y_i}) \quad (2.36)$$

(e.g., cf. [X1]). Note $\Delta = \bar{\Delta} + \check{\Delta}$. For $\ell_1, \ell_2, \ell_3 \in \mathbb{N}$, $0 \leq r < s \leq m+1$ and $\ell \in \overline{0, s-r-1}$, $f \in \check{\mathcal{H}}_{\ell_1, \ell_2}$ and $g \in \check{\mathcal{H}}_{r, m+1-s}$, we have

$$\Delta(\bar{\eta}^{\ell_3} \check{\eta}^\ell f g) = \ell_3(n + \ell_1 + \ell_2 + \ell_3 - 1) \bar{\eta}^{\ell_3-1} \check{\eta}^\ell f g + \ell(\ell + r - s) \bar{\eta}^{\ell_3} \check{\eta}^{\ell-1} f g. \quad (2.37)$$

Suppose $r+1 < s$ and $\ell \in \overline{1, s-r-1}$. If

$$\Delta\left(\sum_{p=0}^{\ell} a_p \bar{\eta}^p \check{\eta}^{\ell-p} f g\right) = 0, \quad (2.38)$$

then

$$(\ell - p)(p + s - r - \ell) a_p = (p+1)(n + \ell_1 + \ell_2 + p) a_{p+1} \quad \text{for } p \in \overline{0, \ell-1}. \quad (2.39)$$

Thus we can take $a_0 = (\ell+1)! [\prod_{\iota_2=1}^{\ell+1} (\iota_2 + n + \ell_1 + \ell_2 - 1)]$ and

$$a_{p+1} = \left[\prod_{\iota_1=0}^p (\ell - \iota_1)(\iota_1 + s - r - \ell) \right] \left[\prod_{\iota_2=p+2}^{\ell+1} \iota_2(\iota_2 + n + \ell_1 + \ell_2 - 1) \right] \quad (2.40)$$

for $p \in \overline{0, \ell-1}$. Denote

$$\begin{aligned} \mathfrak{S}(\ell_1, \ell_2; r, s, \ell) &= (\ell+1)! \left[\prod_{\iota_2=1}^{\ell+1} (\iota_2 + n + \ell_1 + \ell_2 - 1) \right] \tilde{\eta}^\ell + \sum_{p=0}^{\ell-1} \left[\prod_{\iota_1=0}^p (\ell - \iota_1)(\iota_1 + s - r - \ell) \right] \\ &\quad \times \left[\prod_{\iota_2=p+2}^{\ell+1} \iota_2(\iota_2 + n + \ell_1 + \ell_2 - 1) \right] \tilde{\eta}^{p+1} \tilde{\eta}^{\ell-p-1}. \end{aligned} \quad (2.41)$$

For convenience, we treat

$$\mathfrak{S}(\ell_1, \ell_2; r, s, 0) = n + \ell_1 + \ell_2. \quad (2.42)$$

Write

$$\mathcal{G} = \bar{\mathcal{G}} + \check{\mathcal{G}} \cong gl(n, \mathbb{C}) \oplus gl(m, \mathbb{C}). \quad (2.43)$$

Then

$$\mathcal{A} = \bigoplus_{\ell_1, \ell_2, \ell_3=0}^{\infty} \bigoplus_{0 \leq r < s \leq m+1} \bigoplus_{\ell=0}^{s-r-1} (\bar{\eta}^{\ell_1} \bar{\mathcal{H}}_{\ell_2, \ell_3}) (\check{\eta}^\ell \check{\mathcal{H}}_{r, m+1-s}) \quad (2.44)$$

is a direct sum of irreducible \mathcal{G} -submodules. By (2.2),

$$\mathcal{H} = \{f \in \mathcal{A} \mid \Delta(f) = 0\} \quad (2.45)$$

forms a $gl(n|m)$ -submodule, and so it is a \mathcal{G} -submodule. According to (2.37)-(2.42),

$$\mathcal{H} = \bigoplus_{\ell_2, \ell_3=0}^{\infty} \bigoplus_{0 \leq r < s \leq m+1} \bigoplus_{\ell=0}^{s-r-1} \mathfrak{S}(\ell_2, \ell_3; r, s, \ell) (\bar{\mathcal{H}}_{\ell_2, \ell_3} \check{\mathcal{H}}_{r, m+1-s}) \quad (2.46)$$

is a direct sum of irreducible \mathcal{G} -submodules.

For $\ell, \ell' \in \mathbb{N}$, we let

$$\mathcal{A}_{\ell, \ell'} = \sum_{\ell_1, \ell_2 \in \mathbb{N}, \ell_3, \ell_4 \in \overline{0, m}; \ell_1 + \ell_3 = \ell, \ell_2 + \ell_4 = \ell'} \bar{\mathcal{A}}_{\ell_1, \ell_2} \check{\mathcal{A}}_{\ell_3, \ell_4} \quad (2.47)$$

and

$$\mathcal{H}_{\ell, \ell'} = \mathcal{A}_{\ell, \ell'} \cap \mathcal{H}. \quad (2.48)$$

Then $\mathcal{A}_{\ell, \ell'}$ and $\mathcal{H}_{\ell, \ell'}$ are $gl(n|m)$ -submodules. Moreover,

$$\mathcal{H}_{\ell, \ell'} = \bigoplus_{\ell_1+r+\ell_3=\ell, \ell_2+\ell_3+m+1-s=\ell'} \mathfrak{S}(\ell_1, \ell_2; r, s, \ell_3) \bar{\mathcal{H}}_{\ell_1, \ell_2} \check{\mathcal{H}}_{r, m+1-s} \quad (2.49)$$

is a direct sum of irreducible \mathcal{G} -submodules. Take the Cartan subalgebra $H = \bar{H} + \check{H}$ of \mathcal{G} (cf. (2.4)) and the subspace $\mathcal{G}_+ = \bar{\mathcal{G}}_+ + \check{\mathcal{G}}_+$ (cf. (2.5)) spanned by positive root vectors in \mathcal{G} . A \mathcal{G} -singular vector v is a nonzero weight vector of \mathcal{G} such that $\mathcal{G}_+(v) = 0$. We count singular vector up to a nonzero scalar multiple. Hence we have:

Lemma 2.3. *The set*

$$\begin{aligned} &\{\mathfrak{S}(\ell_1, \ell_2; r, s, \ell_3) (x_1^{\ell_1} y_n^{\ell_2} \vec{\theta}_r \vec{\vartheta}_s) \mid \ell_1, \ell_2 \in \mathbb{N}; 0 \leq r < s \leq m+1; \\ &\ell_3 \in \overline{0, s-r-1}; \ell_1 + r + \ell_3 = \ell, \ell_2 + \ell_3 + m + 1 - s = \ell'\} \end{aligned} \quad (2.50)$$

is the set of all the \mathcal{G} -singular vectors in $\mathcal{H}_{\ell, \ell'}$, where $\ell_1 \ell_2 = \ell'_1 \ell'_2 = 0$ if $n = 1$.

Take $H = \bar{H} + \check{H}$ as a Cartan subalgebra of the Lie superalgebra $gl(n|m)$ and

$$gl(n|m)_+ = \mathcal{G}_+ + \sum_{r=1}^n \sum_{s=1}^m \mathbb{C} E_{r, n+s} \quad (2.51)$$

as the subalgebra generated by positive root vectors. A $gl(n|m)$ -singular vector v is a nonzero weight vector of $gl(n|m)$ such that $gl(n|m)_+(v) = 0$. We count singular vector up to a nonzero scalar multiple. Assume that $x_1^{\ell'_1} y_n^{\ell'_2} \vec{\theta}_{r'} \vec{\vartheta}_{s'}$ is a $gl(n|m)$ -singular vector, where $\ell'_1 \ell'_2 = 0$ when $n = 1$. If $r' \neq 0$, then

$$\begin{aligned} E_{1, n+r'}(x_1^{\ell'_1} y_n^{\ell'_2} \vec{\theta}_{r'} \vec{\vartheta}_{s'}) &= (x_1 \partial_{\theta_{r'}} - \vartheta_{r'} \partial_{y_1})(x_1^{\ell'_1} y_n^{\ell'_2} \vec{\theta}_{r'} \vec{\vartheta}_{s'}) \\ &= (-1)^{r'-1} x_1^{\ell'_1+1} y_n^{\ell'_2} \vec{\theta}_{r'-1} \vec{\vartheta}_{s'} - \delta_{1,n} \ell'_2 x_1^{\ell'_1} y_n^{\ell'_2-1} \vartheta_{r'} \vec{\theta}_{r'} \vec{\vartheta}_{s'} \neq 0 \end{aligned} \quad (2.52)$$

by the second equation in (1.12), which contradicts the definition of singular vector. So $r' = 0$. Suppose $\ell'_2 > 0$ and $s' > 1$. Again the second equation in (1.12) gives

$$\begin{aligned} E_{n, n+s'-1}(x_1^{\ell'_1} y_n^{\ell'_2} \vec{\vartheta}_{s'}) &= (x_n \partial_{\theta_{s'-1}} - \vartheta_{s'-1} \partial_{y_n})(x_1^{\ell'_1} y_n^{\ell'_2} \vec{\vartheta}_{s'}) \\ &= -\ell'_2 x_1^{\ell'_1} y_n^{\ell'_2-1} \vec{\theta}_{r'-1} \vec{\vartheta}_{s'-1} \neq 0, \end{aligned} \quad (2.53)$$

which is absurd. Thus $x_1^{\ell'_1} y_n^{\ell'_2} \vec{\theta}_{r'} \vec{\vartheta}_{s'}$ is a $gl(n|m)$ -singular vector if and only if $r' = 0$ and $\ell'_2(s' - 1) = 0$.

For $\ell_1, \ell_2 \in \mathbb{N}$, $1 \leq r < s - 1 \leq m$ and $\ell \in \overline{1, s - r - 1}$,

$$\begin{aligned} E_{1, n+s-1}[\mathfrak{S}(\ell_1, \ell_2; r, s, \ell)] &= (x_1 \partial_{\theta_{s-1}} - \vartheta_{s-1} \partial_{y_1})[\mathfrak{S}(\ell_1, \ell_2; r, s, \ell)] \\ &= x_1 \vartheta_{s-1} \{(\ell + 1)! [\prod_{\iota_2=1}^{\ell+1} (\iota_2 + n + \ell_1 + \ell_2 - 1)] \bar{\eta}^{\ell-1} + \sum_{p=0}^{\ell-2} [\prod_{\iota_1=0}^p (\ell - \iota_1)(\iota_1 + s - r - \ell)] \\ &\quad \times (\ell - p - 1) [\prod_{\iota_2=p+2}^{\ell+1} \iota_2 (\iota_2 + n + \ell_1 + \ell_2 - 1)] \bar{\eta}^{p+1} \bar{\eta}^{\ell-p-2} \\ &\quad - \sum_{p=0}^{\ell-1} (p+1) [\prod_{\iota_1=0}^p (\ell - \iota_1)(\iota_1 + s - r - \ell)] [\prod_{\iota_2=p+2}^{\ell+1} \iota_2 (\iota_2 + n + \ell_1 + \ell_2 - 1)] \bar{\eta}^p \bar{\eta}^{\ell-p-1}\} \\ &= \ell(\ell + 1)(n + \ell + \ell_1 + \ell_2 + r - s) \mathfrak{S}(\ell_1 + 1, \ell_2; r, s - 1, \ell - 1) x_1 \vartheta_{s-1}. \end{aligned} \quad (2.54)$$

Moreover, (2.41) yields

$$\mathfrak{S}(\ell_1, \ell_2; r, s, \ell) = (\ell + 1)! [\prod_{\iota_1=0}^{\ell} (\iota_1 + s - r - \ell)] \eta^{\ell} \text{ if } n + \ell + \ell_1 + \ell_2 + r - s = 0. \quad (2.55)$$

Suppose that $\mathfrak{S}(\ell_1, \ell_2; r, s, \ell_3)(x_1^{\ell_1} y_n^{\ell_2} \vec{\theta}_r \vec{\vartheta}_s)$ is a $gl(n|m)$ -singular vector, then

$$n + \ell_3 + \ell_1 + \ell_2 + r - s = 0 \quad (2.56)$$

and

$$\mathfrak{S}(\ell_1, \ell_2; r, s, \ell_3)(x_1^{\ell_1} y_n^{\ell_2} \vec{\theta}_r \vec{\vartheta}_s) = c \eta^{\ell_3} x_1^{\ell_1} y_n^{\ell_2} \vec{\theta}_r \vec{\vartheta}_s, \quad (2.57)$$

where $c = (\ell_3 + 1)! [\prod_{\iota_1=0}^{\ell} (\iota_1 + s - r - \ell_3)]$ by (2.55). Since

$$E_{i, n+r}(c \eta^{\ell_3} x_1^{\ell_1} y_n^{\ell_2} \vec{\theta}_r \vec{\vartheta}_s) = c \eta^{\ell_3} E_{i, n+r}(x_1^{\ell_1} y_n^{\ell_2} \vec{\theta}_r \vec{\vartheta}_s) \quad (2.58)$$

by (1.12), the arguments in (2.52) and (2.53) show $r = \ell_2(s-1) = 0$. On the other hand, $1 \leq 1+r < s$. So $\ell_2 = 0$. According to (2.56),

$$\ell_3 = s - \ell_1 - n. \quad (2.59)$$

Thus we only get the singular vector

$$\eta^{s-\ell_1-n} x_1^{\ell_1} \vec{\vartheta}_s \in \mathcal{H}_{s-n, m+1-n-\ell_1} \text{ with } s > \ell_1 + n. \quad (2.60)$$

Note $n \leq m$ by $\ell_3 > 0$. Moreover, $s \leq m+1$ implies

$$s - n, m + 1 - n - \ell_1 \leq m + 1 - n. \quad (2.61)$$

Furthermore,

$$(s - n) + (m + 1 - n - \ell_1) = m + 1 - n + \ell_3 > m + 1 - n. \quad (2.62)$$

This shows that

$$\begin{aligned} \mathcal{H}_{\ell, \ell'} \text{ has a unique } gl(n|m)\text{-singular vector if} \\ \ell > m + 1 - n \text{ or } \ell' > m + 1 - n \text{ or } \ell + \ell' \leq m + 1 - n. \end{aligned} \quad (2.63)$$

Suppose $n \leq m$ and $\ell, \ell' \in \overline{0, m+1-n}$ such that $\ell + \ell' > m + 1 - n$. We take

$$s = n + \ell, \quad \ell_1 = m + 1 - n - \ell', \quad \ell_3 = \ell + \ell' + n - m - 1. \quad (2.64)$$

Then

$$\eta^{\ell+\ell'+n-m-1} (x_1^{m+1-n-\ell'} \vec{\vartheta}_{n+\ell}) \in \mathcal{H}_{\ell, \ell'}. \quad (2.65)$$

Hence

$$\mathcal{H}_{\ell, \ell'} \text{ has exactly two } gl(n|m)\text{-singular vectors if } \ell + \ell' > m + 1 - n. \quad (2.66)$$

In summary, we have:

Lemma 2.4. *Let $\ell, \ell' \in \mathbb{N}$. If $\ell > m + 1 - n$ or $\ell' > m + 1 - n$ or $\ell + \ell' \leq m + 1 - n$, the $gl(n|m)$ -module $\mathcal{H}_{\ell, \ell'}$ has a unique $gl(n|m)$ -singular vector. When $\ell, \ell' \leq m + 1 - n$ and $\ell + \ell' > m + 1 - n$, the $gl(n|m)$ -module $\mathcal{H}_{\ell, \ell'}$ has exactly two $gl(n|m)$ -singular vectors.*

Fix $\ell, \ell' \in \mathbb{N}$. Let

$$v_{\ell, \ell'} = x_1^\ell y_n^{\ell_1} \vec{\vartheta}_s, \quad \ell_1 + m + 1 - s = \ell', \quad \ell_1(s-1) = 0, \quad \delta_{n,1} \ell \ell_1 = 0. \quad (2.67)$$

Lemma 2.5. *The $gl(n|m)$ -module $\mathcal{H}_{\ell, \ell'}$ is generated by $v_{\ell, \ell'}$.*

Proof. Let M be the $gl(n|m)$ -submodule of $\mathcal{H}_{\ell, \ell'}$ generated by $v_{\ell, \ell'}$. First consider $n > 1$ or $\ell = 0$. For any $s' \in \overline{s+1, m+1}$, we have

$$E_{n+s'-1, n} E_{n+s'-2, n} \cdots E_{n+s, n} (v_{\ell, \ell'}) = x_1^\ell y_n^{\ell_1+s'-s} \vec{\vartheta}_{s'} \in M \quad (2.68)$$

by (1.13). In other words,

$$\begin{aligned} x_1^\ell y_n^{\ell_2} \vec{\vartheta}_{s'} \in M \text{ for any } \ell_2 \in \mathbb{N} \text{ and } s' \in \overline{1, m+1} \\ \text{such that } \ell_2 + m + 1 - s' = \ell' \text{ and } \delta_{n,1} \ell \ell_2 = 0. \end{aligned} \quad (2.69)$$

If $\ell > 0$, then for any $1 \leq r \leq \min\{\ell, s' - 1\}$, we have

$$E_{n+1,1}E_{n+2,1} \cdots E_{n+r,1}(x_1^\ell y_n^{\ell_2} \vec{\vartheta}_{s'}) = [\prod_{p=0}^{r-1} (\ell - p)] x_1^{\ell-r} y_n^{\ell_2} \vec{\theta}_r \vec{\vartheta}_{s'} \in M \quad (2.70)$$

by (1.13) again. Thus we have showed that

$$x_1^{\ell_3} y_n^{\ell_2} \vec{\theta}_r \vec{\vartheta}_{s'} \in M \text{ whenever } r + \ell_3 = \ell, \ell_2 + m + 1 - s' = \ell', \delta_{n,1} \ell_2 \ell_3 = 0 \quad (2.71)$$

for $\ell_2, \ell_3 \in \mathbb{N}$ and $0 \leq r < s' \leq m + 1$. Recall the Lie algebra \mathcal{G} defined in (2.43). As \mathcal{G} -modules,

$$\sum_{r+\ell_3=\ell, \ell_2+m+1-s'=\ell'} \check{\mathcal{H}}_{\ell_3, \ell_2} \bar{\mathcal{H}}_{r, m+1-s'} \subset M. \quad (2.72)$$

For $i \in \mathbb{N} + 1$, we define

$$\mathcal{H}_{\ell, \ell'}^{(i)} = \{f \in \mathcal{H}_{\ell, \ell'} \mid \bar{\Delta}^i(f) = 0\} \quad (2.73)$$

Then

$$\mathcal{H}_{\ell, \ell'}^{(1)} = \sum_{r+\ell_3=\ell, \ell_2+m+1-s'=\ell'} \check{\mathcal{H}}_{\ell_3, \ell_2} \bar{\mathcal{H}}_{r, m+1-s'} \subset M. \quad (2.74)$$

Denote

$$k_{\ell, \ell'} = \min\{\ell, \ell'\}. \quad (2.75)$$

Then

$$\mathcal{H}_{\ell, \ell'}^{(k_{\ell, \ell'} + 1)} = \mathcal{H}_{\ell, \ell'} \quad (2.76)$$

by (2.49). Let $\ell_1, \ell_2, \ell_3 \in \mathbb{N}$, $0 \leq r' + 1 < s' \leq m + 1$ and $\ell_3 \in \overline{1, s' - r' - 1}$ such that $\ell_1 + \ell_3 + r' = \ell$ and $\ell_2 + \ell_3 + m + 1 - s' = \ell'$. Then

$$\mathfrak{S}(\ell_1, \ell_2; r', s', \ell_3)(x_1^{\ell_1} y_n^{\ell_2} \vec{\theta}_{r'} \vec{\vartheta}_{s'}) \in \mathcal{H}_{\ell, \ell'}^{(\ell_3 + 1)} \quad (2.77)$$

and

$$\bar{\Delta}^{\ell_3} \mathfrak{S}(\ell_1, \ell_2; r', s', \ell_3)(x_1^{\ell_1} y_n^{\ell_2} \vec{\theta}_{r'} \vec{\vartheta}_{s'}) = c(\ell_1, \ell_2; r', s', \ell_3) x_1^{\ell_1} y_n^{\ell_2} \vec{\theta}_{r'} \vec{\vartheta}_{s'} \quad (2.78)$$

with

$$c(\ell_1, \ell_2; r', s', \ell_3) = \ell_3(n + \ell_1 + \ell_3 + \ell_3 - 1)(n + \ell_1 + \ell_2 + \ell_3)(\ell_3 + 1)! \left[\prod_{p=1}^{\ell_3} (s' - r' - p) \right] \quad (2.79)$$

by (2.36) and (2.41). On the other hand,

$$x_1^{\ell_1 + \ell_3} y_n^{\ell_2} \vec{\theta}_{r'} \vec{\vartheta}_{s' - \ell_3} \in M \text{ with } \delta_{n,1} \ell_2 = 0 \quad (2.80)$$

by (2.71) and

$$\begin{aligned} M \ni f &= E_{n+s'-1,1} E_{n+s'-2,1} \cdots E_{n+s'-\ell_3,1} (x_1^{\ell_1 + \ell_3} y_n^{\ell_2} \vec{\theta}_{r'} \vec{\vartheta}_{s' - \ell_3}) \\ &= (-1)^{r' \ell_3} x_1^{\ell_1 + \ell_3} y_1^{\ell_3} y_n^{\ell_2} \vec{\theta}_{r'} \vec{\vartheta}_{s'} + y_n^{\ell_2} \sum_{i=0}^{\ell_3 - 1} \zeta_i y_1^i \end{aligned} \quad (2.81)$$

with $\zeta_0, \dots, \zeta_{\ell_3 - 1} \in \mathbb{C}[x_1] \check{\mathcal{A}}$ (cf. (2.3)). Moreover,

$$\bar{\Delta}^{\ell_3} [(\ell_3!)^2 \binom{\ell_1 + \ell_3}{\ell_3} \mathfrak{S}(\ell_1, \ell_2; r', s', \ell_3)(x_1^{\ell_1} y_n^{\ell_2} \vec{\theta}_{r'} \vec{\vartheta}_{s'}) - (-1)^{r' \ell_3} c(\ell_1, \ell_2; r', s', \ell_3) f] = 0. \quad (2.82)$$

Hence

$$\mathfrak{S}(\ell_1, \ell_2; r', s', \ell_3)(\bar{\mathcal{H}}_{\ell_1, \ell_2} \check{\mathcal{H}}_{r', m+1-s'}) \subset \mathcal{H}_{\ell, \ell'}^{(\ell_3)} + M. \quad (2.83)$$

By (2.49) and induction on i , we have

$$\mathcal{H}_{\ell, \ell'}^{(i)} \subset M \quad \text{for any } i \in \mathbb{N} + 1. \quad (2.84)$$

According to (2.86), $\mathcal{H}_{\ell, \ell'} = M$. So $\mathcal{H}_{\ell, \ell'}$ is generated by $v_{\ell, \ell'}$. \square

Proof of Theorem 1

According to (2.66), a necessary condition for $\mathcal{H}_{\ell, \ell'}$ to be an irreducible $gl(n|m)$ -module is $\ell > m + 1 - n$ or $\ell' > m + 1 - n$ or $\ell + \ell' \leq m + 1 - n$. To prove the sufficiency, we suppose that $\ell > m + 1 - n$ or $\ell' > m + 1 - n$ or $\ell + \ell' \leq m + 1 - n$. Let V be a nonzero $gl(n|m)$ -submodule of $\mathcal{H}_{\ell, \ell'}$. According to Lemma 2.4, $\mathcal{H}_{\ell, \ell'}$ has a unique singular vector $v_{\ell, \ell'}$ (cf. (2.67)). Since V is finite-dimensional, it contains a singular vector. So $v_{\ell, \ell'} \in V$. Lemma 2.5 says $V = \mathcal{H}_{\ell, \ell'}$. Hence $\mathcal{H}_{\ell, \ell'}$ is an irreducible $gl(n|m)$ -module.

Let $\ell_1, \ell_2 \in \mathbb{N}$ and $0 \leq r < s \leq m + 1$ such that $n + \ell_1 + \ell_2 + r - s \geq 0$ and $\delta_{n,1}\ell_1\ell_2 = 0$. For any $\ell_4 \in \mathbb{N} + 1$ and $\ell_3 \in \overline{0, s - r - 1}$,

$$\Delta^{\ell_4+1}(\eta^{\ell_4} \mathfrak{S}(\ell_1, \ell_2; r, s, \ell_3) x_1^{\ell_1} y_n^{\ell_2} \vec{\theta}_r \vec{\vartheta}_s) = 0 \quad (2.85)$$

and

$$\begin{aligned} & \Delta^{\ell_4}(\eta^{\ell_4} \mathfrak{S}(\ell_1, \ell_2; r, s, \ell_3) x_1^{\ell_1} y_n^{\ell_2} \vec{\theta}_r \vec{\vartheta}_s) \\ &= \ell_4! \left[\prod_{i=1}^{\ell_4} (n + i + \ell_1 + 2\ell_3 + r - s) \right] \mathfrak{S}(\ell_1, \ell_2; r, s, \ell_3) (x_1^{\ell_1} y_n^{\ell_2} \vec{\theta}_r \vec{\vartheta}_s) \neq 0 \end{aligned} \quad (2.86)$$

by (2.27) and (2.36). Thus the set

$$\{\eta^{\ell_4} \mathfrak{S}(\ell_1, \ell_2; r, s, \ell_3) x_1^{\ell_1} y_n^{\ell_2} \vec{\theta}_r \vec{\vartheta}_s \mid \ell_4 \in \mathbb{N}, \ell_3 \in \overline{0, s - r - 1}\} \quad (2.87)$$

is linearly independent.

Note that

$$\check{\eta}^{s-r} x_1^{\ell_1} y_n^{\ell_2} \vec{\theta}_r \vec{\vartheta}_s = 0 \quad (2.88)$$

by (2.19) and (2.20). So for any $k \in \mathbb{N}$,

$$\begin{aligned} & \text{Span}\{\eta^{\ell_4} \mathfrak{S}(\ell_1, \ell_2; r, s, \ell_3) x_1^{\ell_1} y_n^{\ell_2} \vec{\theta}_r \vec{\vartheta}_s \mid \ell_4 \in \mathbb{N}, \ell_3 \in \overline{0, s - r - 1}; \ell_3 + \ell_4 = k\} \\ & \subset \text{Span}\{\check{\eta}^{\ell_5} \check{\eta}^{\ell_6} x_1^{\ell_1} y_n^{\ell_2} \vec{\theta}_r \vec{\vartheta}_s \mid \ell_5 \in \mathbb{N}; \ell_6 \in \overline{0, s - r - 1}; \ell_5 + \ell_6 = k\}. \end{aligned} \quad (2.89)$$

But the linear independency of (2.87) implies that the above subspaces have the same dimension. Thus

$$\begin{aligned} & \text{Span}\{\eta^{\ell_4} \mathfrak{S}(\ell_1, \ell_2; r, s, \ell_3) x_1^{\ell_1} y_n^{\ell_2} \vec{\theta}_r \vec{\vartheta}_s \mid \ell_4 \in \mathbb{N}, \ell_3 \in \overline{0, s - r - 1}\} \\ &= \text{Span}\{\check{\eta}^{\ell_5} \check{\eta}^{\ell_6} x_1^{\ell_1} y_n^{\ell_2} \vec{\theta}_r \vec{\vartheta}_s \mid \ell_5 \in \mathbb{N}; \ell_6 \in \overline{0, s - r - 1}\}. \end{aligned} \quad (2.90)$$

Therefore, as \mathcal{G} -modules,

$$\bigoplus_{\ell_3=0}^{s-r-1} \sum_{\ell_4=0}^{\infty} \eta^{\ell_4} \mathfrak{S}(\ell_1, \ell_2; r, s, \ell_3) \bar{\mathcal{H}}_{\ell_1, \ell_2} \check{\mathcal{H}}_{r, m+1-s} = \bigoplus_{\ell_5=0}^{s-r-1} \sum_{\ell_6=0}^{\infty} \check{\eta}^{\ell_5} \check{\eta}^{\ell_6} \bar{\mathcal{H}}_{\ell_1, \ell_2} \check{\mathcal{H}}_{r, m+1-s}. \quad (2.91)$$

Assume that $|\ell - \ell'| > m + 1 - n$ or $\ell + \ell' \leq m + 1 - n$. According to (2.44) and (2.90), the \mathcal{G} -module

$$\begin{aligned} \mathcal{A}_{\ell, \ell'} &= \text{Span}\{\bar{\eta}^{\ell_5} \check{\eta}^{\ell_6} \bar{\mathcal{H}}_{\ell_1, \ell_2} \check{\mathcal{H}}_{r, m+1-s} \mid \ell_1, \ell_2, \ell_6 \in \mathbb{N}; 0 \leq r < s \leq m+1; \ell_5 \in \overline{0, s-r-1}; \\ &\quad \delta_{n,1} \ell_1 \ell_2 = 0; \ell_1 + \ell_5 + \ell_6 + r = \ell, \ell_2 + \ell_5 + \ell_6 + m + 1 - s = \ell'\} \\ &= \text{Span}\{\eta^{\ell_4} \mathfrak{S}(\ell_1, \ell_2; r, s, \ell_3) \bar{\mathcal{H}}_{\ell_1, \ell_2} \check{\mathcal{H}}_{r, m+1-s} \mid \ell_1, \ell_2, \ell_6 \in \mathbb{N}; \delta_{n,1} \ell_1 \ell_2 = 0; 0 \leq r < s \leq m+1; \\ &\quad \ell_5 \in \overline{0, s-r-1}; \ell_1 + \ell_3 + \ell_4 + r = \ell, \ell_2 + \ell_3 + \ell_4 + m + 1 - s = \ell'\}. \end{aligned} \quad (2.92)$$

According to (2.49), (2.91) and (2.92),

$$\mathcal{A}_{\ell, \ell'} = \bigoplus_{i=0}^{k_{\ell, \ell'}} \eta^i \mathcal{H}_{\ell-i, \ell'-i} \quad (2.94)$$

(cf. (2.75)). Let $i \in \overline{0, k_{\ell, \ell'}}$. If $|\ell - \ell'| > m + 1 - n$, then

$$\ell - i \geq |\ell - \ell'| > m + 1 - n \quad \text{or} \quad \ell' - i \geq |\ell - \ell'| > m + 1 - n. \quad (2.95)$$

When $\ell + \ell' \leq m + 1 - n$,

$$(\ell - i) + (\ell' - i) = \ell + \ell' - 2i \leq m + 1 - n. \quad (2.96)$$

Thus all $\mathcal{H}_{\ell-i, \ell'-i}$ are irreducible $gl(n|m)$ -submodules. Hence (2.94) is a direct sum of irreducible $gl(n|m)$ -submodules.

This completes the proof of Theorem 1. \square

The following result will be used to obtain explicit bases for modules.

Lemma 2.6 (Xu [X1]). *Let \mathcal{B} be a commutative associative algebra and let \mathcal{A} be a free \mathcal{B} -module generated by a filtrated subspace $V = \bigcup_{r=0}^{\infty} V_r$ (i.e., $V_r \subset V_{r+1}$). Let T_1 be a linear operator on \mathcal{A} with a right inverse T_1^- such that*

$$T_1(\mathcal{B}), T_1^-(\mathcal{B}) \subset \mathcal{B}, \quad T_1(\eta_1 \eta_2) = T_1(\eta_1) \eta_2, \quad T_1^-(\eta_1 \eta_2) = T_1^-(\eta_1) \eta_2 \quad (2.97)$$

for $\eta_1 \in \mathcal{B}$, $\eta_2 \in V$, and let T_2 be a linear operator on \mathcal{A} such that

$$T_2(V_{r+1}) \subset \mathcal{B}V_r, \quad T_2(f\zeta) = fT_2(\zeta) \quad \text{for } r \in \mathbb{N}, \quad f \in \mathcal{B}, \quad \zeta \in \mathcal{A}. \quad (2.98)$$

Then we have

$$\begin{aligned} &\{g \in \mathcal{A} \mid (T_1 + T_2)(g) = 0\} \\ &= \text{Span}\left\{\sum_{i=0}^{\infty} (-T_1^- T_2)^i (hg) \mid g \in V, h \in \mathcal{B}; T_1(h) = 0\right\}. \end{aligned} \quad (2.99)$$

Set

$$\epsilon_i = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0) \in \mathbb{N}^n. \quad (2.100)$$

For each $i \in \overline{1, n}$, we define the linear operator $\int_{(x_i)}$ on \mathcal{A} by:

$$\int_{(x_i)} (x^\alpha) = \frac{x^{\alpha + \epsilon_i}}{\alpha_i + 1} \quad \text{for } \alpha \in \mathbb{N}^n. \quad (2.101)$$

Furthermore, we let

$$\int_{(x_i)}^{(0)} = 1, \quad \int_{(x_i)}^{(m)} = \overbrace{\int_{(x_i)} \cdots \int_{(x_i)}}^m \quad \text{for } 0 < m \in \mathbb{Z} \quad (2.102)$$

and denote

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}, \quad \int^{(\alpha)} = \int_{(x_1)}^{(\alpha_1)} \int_{(x_2)}^{(\alpha_2)} \cdots \int_{(x_n)}^{(\alpha_n)} \quad \text{for } \alpha \in \mathbb{N}^n. \quad (2.103)$$

Obviously, $\int^{(\alpha)}$ is a right inverse of ∂^α for $\alpha \in \mathbb{N}^n$. We remark that $\int^{(\alpha)} \partial^\alpha \neq 1$ if $\alpha \neq 0$ due to $\partial^\alpha(1) = 0$. In this paper, our T_1 's are of the type ∂^α and the right inverse $T_1^- = \int^{(\alpha)}$.

Denote $\Gamma_0 = \emptyset$ and

$$\Gamma_\ell = \{\vec{j} = (j_1, j_2, \dots, j_\ell) \mid 1 \leq j_1 < j_2 < \cdots < j_\ell \leq m\} \quad \text{for } \ell \in \overline{1, m}. \quad (2.104)$$

Moreover, we set

$$\theta_\emptyset = \vartheta_\emptyset = 1, \quad \theta_{\vec{j}} = \theta_{j_1} \theta_{j_2} \cdots \theta_{j_\ell}, \quad \vartheta_{\vec{j}} = \vartheta_{j_1} \vartheta_{j_2} \cdots \vartheta_{j_\ell}. \quad (2.105)$$

Then the set

$$\left\{ \sum_{i=0}^{\infty} \frac{(-1)^i x_1^i y_1^i}{\prod_{r=1}^i (\alpha_1 + i)(\beta_1 + i)} \left(\sum_{s=2}^n \partial_{x_2} \partial_{y_2} + \sum_{r=1}^m \partial_{\theta_r} \partial_{\vartheta_r} \right)^i (x^\alpha y^\beta \theta_{\vec{j}} \vartheta_{\vec{k}}) \mid \alpha, \beta \in \mathbb{N}^n; \right. \\ \left. \vec{j} \in \Gamma_{\ell_1}; \vec{k} \in \Gamma_{\ell_2}; \alpha_1 \beta_1 = 0; \ell_1, \ell_2 \in \overline{0, m}; |\alpha| + \ell_1 = \ell; |\beta| + \ell_2 = \ell' \right\} \quad (2.106)$$

forms a basis of $\mathcal{H}_{\ell, \ell'}$ by Lemma 2.6 with $T_1 = \partial_{x_1} \partial_{y_1}$, $T_2 = \Delta - T_1$ and $T_1^- = \int_{(y_1)}$.

Remark 2.7. If $\ell, \ell' \leq m+1-n$ and $\ell + \ell' > m+1-n$, then $\mathcal{H}_{\ell, \ell'}$ is an indecomposable $gl(n|m)$ -module by (2.66) and Lemma 2.5, and $\mathcal{H}_{\ell, \ell'} \cap \eta \mathcal{A}_{\ell-1, \ell'-1} \neq \{0\}$. This also shows that $\mathcal{A}_{\ell, \ell'}$ is not completely reducible when $|\ell - \ell'| \leq m+1-n$ and $\ell + \ell' > m+1-n$.

3 Proof of Theorem 2

In this section, we want to prove Theorem 2. Recall the settings in (1.5)-(1.11) and (1.18)-(1.25).

The Laplace operator in (2.6) changes to

$$\bar{\Delta} = - \sum_{i=1}^{n_1} x_i \partial_{y_i} + \sum_{r=n_1+1}^{n_2} \partial_{x_r} \partial_{y_r} - \sum_{s=n_2+1}^n y_s \partial_{x_s} \quad (3.1)$$

and its dual changes to

$$\bar{\eta} = \sum_{i=1}^{n_1} y_i \partial_{x_i} + \sum_{r=n_1+1}^{n_2} x_r y_r + \sum_{s=n_2+1}^n x_s \partial_{y_s}. \quad (3.2)$$

We take (2.19) and then supersymmetric Laplace operator in (1.23) and its dual in (1.24) can be written as:

$$\Delta = \bar{\Delta} + \check{\Delta}, \quad \eta = \bar{\eta} + \check{\eta}. \quad (3.3)$$

Then with respect to the representation in (1.18)-(1.22), we have

$$E_{i,j}\Delta = \Delta E_{i,j}, \quad E_{i,j}\eta = \eta E_{i,j} \quad \text{for } i, j \in \overline{1, m+n}. \quad (3.4)$$

Moreover, we take the settings in (2.1), (2.3)-(2.5) and (2.7).

Denote

$$\bar{\mathcal{A}}_{\langle \ell_1, \ell_2 \rangle} = \text{Span}\{x^\alpha y^\beta \mid \alpha, \beta \in \mathbb{N}^n; \sum_{r=n_1+1}^n \alpha_r - \sum_{i=1}^{n_1} \alpha_i = \ell_1; \sum_{i=1}^{n_2} \beta_i - \sum_{r=n_2+1}^n \beta_r = \ell_2\} \quad (3.5)$$

for $\ell_1, \ell_2 \in \mathbb{Z}$. Then $\bar{\mathcal{A}}_{\langle \ell_1, \ell_2 \rangle}$ forms a $\bar{\mathcal{G}}$ -submodule. Moreover, for $\ell, \ell' \in \mathbb{Z}$, we let

$$\mathcal{A}_{\langle \ell, \ell' \rangle} = \sum_{\ell_1, \ell_2 \in \mathbb{Z}, \ell_3, \ell_4 \in \overline{0, m}; \ell_1 + \ell_3 = \ell, \ell_2 + \ell_4 = \ell'} \bar{\mathcal{A}}_{\langle \ell_1, \ell_2 \rangle} \check{\mathcal{A}}_{\ell_3, \ell_4}. \quad (3.6)$$

It can be verified that $\mathcal{A}_{\langle \ell, \ell' \rangle}$ forms a $gl(n|m)$ -submodule. Define

$$\mathcal{H} = \{f \in \mathcal{A} \mid \Delta(f) = 0\}, \quad \mathcal{H}_{\langle \ell_1, \ell_2 \rangle} = \mathcal{H} \bigcap \mathcal{A}_{\langle \ell_1, \ell_2 \rangle}. \quad (3.7)$$

By (3.4), $\mathcal{H}_{\langle \ell, \ell' \rangle}$ forms a $gl(n|m)$ -submodule of $\mathcal{A}_{\langle \ell, \ell' \rangle}$. According to [LX], we have:

Lemma 3.1. *The nonzero vectors in*

$$\{\mathbb{C}[\bar{\eta}](x_i^{m_1} y_j^{m_2}) \mid m_1, m_2 \in \mathbb{N}; i = n_1, n_1 + 1; j = n_2, n_2 + 1 - \delta_{n_2, n}\} \quad (3.8)$$

are all the singular vectors of $\bar{\mathcal{G}}$ (cf. (2.3)-(2.5)) in $\bar{\mathcal{A}}$ (cf. (2.1)).

Recall $\mathcal{G} = \bar{\mathcal{G}} + \check{\mathcal{G}}$ (cf. (2.3)). Take the Cartan subalgebra $H = \bar{H} + \check{H}$ of \mathcal{G} (cf. (2.4)) and the subspace $\mathcal{G}_+ = \bar{\mathcal{G}}_+ + \check{\mathcal{G}}_+$ (cf. (2.5)) spanned by positive root vectors in \mathcal{G} . Then the nonzero vectors in

$$\{\mathbb{C}[\bar{\eta}, \check{\eta}](x_i^{m_1} y_j^{m_2} \vec{\theta}_r \vec{\vartheta}_s) \mid m_1, m_2 \in \mathbb{N}; i = n_1, n_1 + 1; j = n_2, n_2 + 1 - \delta_{n_2, n}; 0 \leq r < s \leq m + 1\} \quad (3.9)$$

are all the \mathcal{G} -singular vectors in \mathcal{A} . Choose H as a Cartan subalgebra of the Lie superalgebra $gl(n|m)$ and $gl(n|m)_+ = \mathcal{G}_+ + \sum_{r=1}^n \sum_{s=1}^m \mathbb{C}E_{r, n+s}$ as the subalgebra generated by positive root vectors.

Fix $\ell, \ell' \in \mathbb{Z}$. Then a $gl(n|m)$ -singular vector in $\mathcal{A}_{\langle \ell, \ell' \rangle}$ must be of the form

$$f = \sum_{p=0}^{\min\{s-r-1, \ell_1\}} b_p \bar{\eta}^{\ell_1-p} \check{\eta}^p (x_i^{m_1} y_j^{m_2} \vec{\theta}_r \vec{\vartheta}_s), \quad (3.10)$$

where $\ell_1, m_1, m_2 \in \mathbb{N}$, $0 \leq r < s \leq m + 1$, $b_p \in \mathbb{C}$ and

$$(i, j) \in \{(n_1, n_2), (n_1, n_2 + 1 - \delta_{n_2, n}), (n_1 + 1, n_2), (n_1 + 1, n_2 + 1 - \delta_{n_2, n})\}. \quad (3.11)$$

Suppose that $\ell_1 = 0$ or $s - r - 1 = 0$. Then we can assume $f = x_i^{m_1} y_j^{m_2} \vec{\theta}_r \vec{\vartheta}_s$. If $r \neq 0$, then (1.21) implies

$$E_{n_1+1, n+r}(f) = (-1)^{r-1} x_{n_1+1} x_i^{m_1} y_j^{m_2} \vec{\theta}_{r-1} \vec{\vartheta}_s \neq 0, \quad (3.12)$$

which is absurd. So $r = 0$. Assume that $m_2 > 0$, $s > 1$ and $j = n_2$. According to (1.21),

$$E_{n_2, n_2+1}(f) = -m_2 x_i^{m_1} y_j^{m_2-1} \vartheta_1 \vec{\vartheta}_s \neq 0, \quad (3.13)$$

which leads a contradiction. Hence $m_2(s-1) = 0$ if $j = n_2$. If $n_2 < n$, then we have

$$E_{n_2+1, n_2+1}(f) = x_i^{m_1} y_j^{m_2} y_{n_2+1} \vartheta_1 \vec{\vartheta}_s \neq 0 \quad (3.14)$$

by (1.21), which is absurd. Thus $s = 1$. In summary,

$$f = x_i^{m_1} y_j^{m_2} \vec{\vartheta}_s \quad \text{with } s = 1 \text{ or } n_2 = n \text{ and } m_2 = 0. \quad (3.15)$$

Consider the case $\ell_1 > 0$ and $s-r-1 > 0$. By (1.21) and the fact $\check{\eta}^{s-r-1} \vartheta_{r+1} \vec{\theta}_r \vec{\vartheta}_s = 0$, we have

$$\begin{aligned} 0 &= E_{n_1+1, n+r+1}(f) = (x_{n_1+1} \partial_{\theta_{r+1}} - \vartheta_{r+1} \partial_{y_{n_1+1}})(f) \\ &= \left[\sum_{p=1}^{\min\{s-r-1, \ell_1\}} p b_p \bar{\eta}^{\ell_1-p} \check{\eta}^{p-1} - \sum_{p=0}^{\min\{s-r-1, \ell_1\}-1} (\ell_1 - p) b_p \bar{\eta}^{\ell_1-p-1} \check{\eta}^p (x_i^{m_1} y_j^{m_2} \vec{\theta}_r \vec{\vartheta}_s) \right] \\ &\quad \times x_{n_1+1} \vartheta_{r+1} x_i^{m_1} y_j^{m_2} \vec{\theta}_r \vec{\vartheta}_s, \end{aligned} \quad (3.16)$$

which implies

$$f = b_0 (\bar{\eta} + \check{\eta})^{\ell_1} (x_i^{m_1} y_j^{m_2} \vec{\theta}_r \vec{\vartheta}_s) = b_0 \eta^{\ell_1} (x_i^{m_1} y_j^{m_2} \vec{\theta}_r \vec{\vartheta}_s). \quad (3.17)$$

The arguments in the previous paragraph and (3.4) give:

Lemma 3.2. *Any $gl(n|m)$ -singular vector in $\mathcal{A}_{\langle \ell, \ell' \rangle}$ must be of the form:*

$$\eta^{\ell_1} (x_i^{m_1} y_j^{m_2} \vec{\vartheta}_s) \quad \text{with } \ell_1, m_1, m_2 \in \mathbb{N}, s \in \overline{1, m+1} \text{ and (3.10)} \quad (3.18)$$

such that $s = 1$ or $n_2 = n$ and $m_2 = 0$.

We define

$$\flat = \sum_{r=n_1+1}^n x_r \partial_{x_r} - \sum_{i=1}^{n_1} x_i \partial_{x_i}, \quad \flat' = \sum_{i=1}^{n_2} y_i \partial_{y_i} - \sum_{r=n_2+1}^n y_r \partial_{y_r}. \quad (3.19)$$

Then

$$\bar{\mathcal{A}}_{\langle \ell, \ell' \rangle} = \{f \in \bar{\mathcal{A}} \mid \flat(f) = \ell f; \flat'(f) = \ell' f\}. \quad (3.20)$$

We calculate

$$\bar{\Delta} \bar{\eta} = \bar{\eta} \bar{\Delta} + n_2 - n_1 + \flat + \flat'. \quad (3.21)$$

For $\ell_1 \in \mathbb{N} + 1$ and $f \in \mathcal{H}_{\langle \ell, \ell' \rangle}$ (cf. (3.7)), (2.27) and (3.21) imply

$$\Delta \eta^{\ell_1}(f) = \ell_1 (n_2 - n_1 - m + \ell + \ell' + \ell_1 - 1) \eta^{\ell_1-1}(f). \quad (3.22)$$

Thus

$$\Delta \eta^{\ell_1}(f) = 0 \iff \ell + \ell' \leq n_1 + m - n_2 \quad \text{and} \quad \ell_1 = n_1 + m - n_2 - \ell - \ell' + 1. \quad (3.23)$$

If the condition holds, then

$$\eta^{\ell_1}(f) \in \mathcal{H}_{\langle n_1+m-n_2-\ell'+1, n_1+m-n_2-\ell+1 \rangle}. \quad (3.24)$$

Moreover,

$$(n_1 + m - n_2 - \ell' + 1) + (n_1 + m - n_2 - \ell + 1) \geq n_1 + m - n_2 + 2. \quad (3.25)$$

Observe that

$$x_{n_1}^{m_1} y_{n_2}^{m_2} \vec{\vartheta}_s \in \mathcal{H}_{\langle -m_1, m+1+m_2-s \rangle}, \quad x_{n_1}^{m_1} y_{n_2+1}^{m_2} \vec{\vartheta}_1 \in \mathcal{H}_{\langle -m_1, m-m_2 \rangle}, \quad (3.27)$$

$$x_{n_1+1}^{m_1} y_{n_2}^{m_2} \vec{\vartheta}_s \in \mathcal{H}_{\langle m_1, m+1+m_2-s \rangle}, \quad x_{n_1+1}^{m_1} y_{n_2+1}^{m_2} \vec{\vartheta}_1 \in \mathcal{H}_{\langle m_1, m-m_2 \rangle}. \quad (3.28)$$

By Lemma 3.2,

$$\text{any nonzero } \mathcal{H}_{\langle \ell, \ell' \rangle} \text{ contains a singular vector of the form } x_i^{m_1} y_j^{m_2} \vec{\vartheta}_s, \quad (3.29)$$

where $s = 1$ or $n_2 = n$ and $m_2 = 0$.

Now we consider $f = x_i^{m_1} y_j^{m_2} \vec{\vartheta}_s$ with $m_1, m_2 \in \mathbb{N}$, $s \in \overline{1, m+1}$ and (3.11) such that $s = 1$ or $n_2 = n$ and $m_2 = 0$. Assume $\Delta \eta^{\ell_1}(f) = 0$ for some $\ell_1 \in \mathbb{N} + 1$.

Case 1. $(i, j) = (n_1, n_2)$.

In this subcase, $\ell = -m_1$ and $\ell' = m_2 + m + 1 - s$ by (3.27). Thus $m_2 - m_1 + 1 - s \leq n_1 - n_2$ and $\ell_1 = n_1 + m_1 + s - n_2 - m_2$ by (3.23). So

$$\eta^{\ell_1}(f) \in \mathcal{H}_{\langle n_1+s-n_2-m_2, n_1+m_1-n_2+m+1 \rangle}. \quad (3.30)$$

Case 2. $(i, j) = (n_1, n_2 + 1)$.

In this subcase $s = 1$, $\ell = -m_1$ and $\ell' = m - m_2$ by (3.27). Thus $m_1 + m_2 \geq n_2 - n_1$ and $\ell_1 = n_1 + m_1 + m_2 - n_2 + 1$ by (3.23). Hence

$$\eta^{\ell_1}(f) \in \mathcal{H}_{\langle n_1+m_2-n_2+1, n_1+m_1-n_2+m+1 \rangle}. \quad (3.31)$$

Case 3. $(i, j) = (n_1 + 1, n_2)$.

In this subcase, $\ell = m_1$ and $\ell' = m_2 + m + 1 - s$ by (3.28). Thus $m_2 + m_1 + 1 - s \leq n_1 - n_2$ and $\ell_1 = n_1 - m_1 + s - n_2 - m_2$ by (3.23). So

$$\eta^{\ell_1}(f) \in \mathcal{H}_{\langle n_1+s-n_2-m_2, n_1-m_1-n_2+m+1 \rangle}. \quad (3.32)$$

Case 4. $(i, j) = (n_1 + 1, n_2 + 1)$.

In this subcase $s = 1$, $\ell = m_1$ and $\ell' = m - m_2$ by (3.28). Thus $m_1 - m_2 \leq n_1 - n_2$ and $\ell_1 = n_1 - m_1 + m_2 - n_2 + 1$ by (3.23). Hence

$$\eta^{\ell_1}(f) \in \mathcal{H}_{\langle n_1+m_2-n_2+1, n_1-m_1-n_2+m+1 \rangle}. \quad (3.33)$$

Thus we obtain:

Lemma 3.3. *A nonzero $gl(n|m)$ -module $\mathcal{H}_{\langle \ell, \ell' \rangle}$ has a unique singular vector if and only if $\ell + \ell' \leq n_1 + m + 1 - n_2$ or $\ell \notin \overline{n_1 + 1 - n, n_1 + m + 1 - n}$ and $n_2 = n$. If the condition holds, the unique singular vector is of the form $x_i^{m_1} y_j^{m_2} \vec{\vartheta}_s$ with (3.11), where $s = 1$ or $n_2 = n$ and $m_2 = 0$.*

Fix $\mathcal{H}_{\langle \ell, \ell' \rangle} \neq \{0\}$. Assume

$$v_{\ell, \ell'} = x_i^{m_1} y_j^{m_2} \vec{\vartheta}_s \in \mathcal{H}_{\langle \ell, \ell' \rangle} \quad (3.34)$$

for some (i, j) in (3.11), $m_1, m_2 \in \mathbb{N}$ and $s \in \overline{1, m+1}$ such that $s = 1$ or $n_2 = n$ and $m_2 = 0$.

Lemma 3.4. *As a $gl(n|m)$ -module, $\mathcal{H}_{\langle \ell, \ell' \rangle}$ is generated by $v_{\ell, \ell'}$.*

Proof. Denote

$$\tilde{\Gamma} = \{\tilde{\alpha} = (\alpha_{n_1+1}, \dots, \alpha_{n_2}) \in \mathbb{N}^{n_2-n_1}\}, \quad |\tilde{\alpha}| = \sum_{i=1}^{n_2-n_1} \alpha_{n_1+i}. \quad (3.35)$$

Set

$$\tilde{\mathcal{A}} = \check{\mathcal{A}}[x_{n_1+1}, \dots, x_{n_2}, y_{n_1+1}, \dots, y_{n_2}], \quad (3.36)$$

$$\tilde{\mathcal{H}}_{\langle \ell_1, \ell_2 \rangle} = \tilde{\mathcal{A}} \cap \mathcal{H}_{\langle \ell_1, \ell_2 \rangle}, \quad (3.37)$$

$$\tilde{\Delta} = - \sum_{i=1}^{n_1} x_i \partial_{y_i} + \sum_{r=n_1+2}^{n_2} \partial_{x_r} \partial_{y_r} - \sum_{s=n_2+1}^n y_s \partial_{x_s} + \check{\Delta}. \quad (3.38)$$

Then $\Delta = \partial_{x_{n_1+1}} \partial_{y_{n_1+1}} + \tilde{\Delta}$. For any $k_1, k_2 \in \mathbb{N}$, we define the operator

$$T_{k_1, k_2} = \sum_{i=0}^{\infty} \frac{(-1)^i x_{n_1+1}^{k_1+i} y_{n_1+1}^{k_2+i}}{\prod_{r=1}^i (k_1 + r)(k_2 + r)} \tilde{\Delta}^i. \quad (3.39)$$

Then Lemma 2.6 yields

$$\begin{aligned} \mathcal{H}_{\langle \ell, \ell' \rangle} &= \text{Span}\{T_{\alpha_{n_1+1}, \beta_{n_1+1}}([\prod_{i \neq n_1+1} x_i^{\alpha_i} y_i^{\beta_i}])\theta_{\vec{j}} \vec{\vartheta}_{\vec{k}} \mid \alpha, \beta \in \mathbb{N}^n; \\ &\quad \vec{j} \in \Gamma_{\ell_1}; \vec{k} \in \Gamma_{\ell_2}; \ell_1, \ell_2 \in \overline{0, m}; \alpha_{n_1+1} \beta_{n_1+1} = 0; \\ &\quad \sum_{s=n_1+1}^n \alpha_s - \sum_{r=1}^{n_1} \alpha_r + \ell_1 = \ell; \sum_{r=1}^{n_2} \beta_r - \sum_{s=n_2+1}^n \beta_s + \ell_2 = \ell'\}, \end{aligned} \quad (3.40)$$

$$\begin{aligned} \tilde{\mathcal{H}}_{\langle \ell, \ell' \rangle} &= \text{Span}\{T_{\alpha_{n_1+1}, \beta_{n_1+1}}([\prod_{i=n_1+2}^{n_2} x_i^{\alpha_i} y_i^{\beta_i}])\theta_{\vec{j}} \vec{\vartheta}_{\vec{k}} \mid \tilde{\alpha}, \tilde{\beta} \in \tilde{\Gamma}; \vec{j} \in \Gamma_{\ell_1}; \vec{k} \in \Gamma_{\ell_2}; \\ &\quad \alpha_{n_1+1} \beta_{n_1+1} = 0; \ell_1, \ell_2 \in \overline{0, m}; |\tilde{\alpha}| + \ell_1 = \ell; |\tilde{\beta}| + \ell_2 = \ell'\}. \end{aligned} \quad (3.41)$$

Write

$$\mathcal{G}_1 = \sum_{i,j=1}^{n_1} \mathbb{C} E_{i,j}, \quad \mathcal{G}_2 = \sum_{r,s=n_2+1}^n \mathbb{C} E_{r,s}, \quad (3.42)$$

$$\mathcal{G}_3 = \sum_{n_1+1 \neq i, j \in \overline{1, n_2}} \mathbb{C} E_{i,j}, \quad \mathcal{G}_4 = \sum_{r,s=n_1+2}^n \mathbb{C} E_{r,s}. \quad (3.43)$$

Then

$$\xi \tilde{\Delta} = \tilde{\Delta} \xi \quad \text{for } \xi \in \mathcal{G}_i, \ i \in \overline{1, 4}. \quad (3.44)$$

Denote by V the $gl(n|m)$ -submodule of $\mathcal{H}_{\langle \ell, \ell' \rangle}$ generated by $v_{\ell, \ell'}$. By (1.18)-(1.20)

$$-E_{n_1+1, n_1}|_{\mathcal{A}} = x_{n_1} x_{n_1+1} + y_{n_1} \partial_{y_{n_1+1}}, \quad E_{n_2+1, n_2}|_{\mathcal{A}} = x_{n_2+1} \partial_{x_{n_2}} + y_{n_2} y_{n_2+1}. \quad (3.45)$$

According to (1.21) and (1.22),

$$E_{n_2, n+r} = -x_{n_2} \partial_{\theta_r} + \vartheta_r \partial_{y_{n_2}}, \quad E_{n+r, n_2} = \theta_r \partial_{x_{n_2}} + y_{n_2} \partial_{\vartheta_r} \quad \text{for } r \in \overline{1, m}. \quad (3.46)$$

Repeatedly applying the operators in (3.45) and (3.46) to $v_{\ell, \ell'}$, we obtain

$$x_{n_1}^{p_1} x_{n_1+1}^{p_2} y_{n_2}^{p_3} y_{n_2+1}^{p_4} \vec{\vartheta}_{s'} \in V \quad (3.47)$$

for $p_i \in \mathbb{N}$ and $s' \in \overline{1, m+1}$ such that

$$p_2 - p_1 = \ell, \quad p_3 - p_4 + m + 1 - s' = \ell', \quad p_3(s' - 1) = 0. \quad (3.48)$$

Lemma 2.5, (2.67), and (3.37) tell us that

$$x_{n_1}^{p_1} y_{n_2+1}^{p_4} \tilde{\mathcal{H}}_{\langle p_2, p_3+m+1-s' \rangle} \subset V. \quad (3.49)$$

Let $U(\mathcal{G}_i)$ be the universal enveloping of the Lie algebra \mathcal{G}_i . Applying $U(\mathcal{G}_1)$ and $U(\mathcal{G}_2)$ to (3.49), we get

$$\left[\prod_{\iota_1=1}^{n_1} x_{\iota_1}^{\alpha_{\iota_1}} \right] \left[\prod_{\iota_2=n_2+1}^n y_{\iota_2}^{\beta_{\iota_2}} \right] \tilde{\mathcal{H}}_{\langle p_2, p_3+m+1-s' \rangle} \subset V \quad (3.50)$$

for any $(\alpha_1, \dots, \alpha_{n_1}) \in \mathbb{N}^{n_1}$ and $(\beta_{n_2+1}, \dots, \beta_n) \in \mathbb{N}^{n-n_2}$ such that $\sum_{\iota_1=1}^{n_1} \alpha_{\iota_1} = p_1$ and $\sum_{\iota_2=n_2+1}^n \beta_{\iota_2} = p_4$. Applying $U(\mathcal{G}_3)$ to (3.50), we obtain

$$T_{\alpha_{n_1+1}, \beta_{n_1+1}} \left(\left[\prod_{i \neq n_1+1} x_i^{\alpha_i} y_i^{\beta_i} \right] \theta_{\vec{j}} \vartheta_{\vec{k}} \right) \in V \quad (3.51)$$

by (3.41) and (3.44), where $\alpha, \beta, \vec{j}, \vec{k}$ are as those in (3.40) and $\alpha_{n_2+1} = \dots = \alpha_n = 0$. Finally, we get (3.51) with any $\alpha, \beta, \vec{j}, \vec{k}$ in (3.40) by (3.44) and applying $U(\mathcal{G}_4)$. According to (3.40), $V = \mathcal{H}_{\langle \ell, \ell' \rangle}$. \square

Proof of Theorem 2

Suppose $\ell + \ell' \leq n_1 + m + 1 - n_2$ or $\ell \notin \overline{n_1 + 1 - n, n_1 + m + 1 - n}$ and $n_2 = n$. Let V be a nonzero submodule of $\mathcal{H}_{\langle \ell, \ell' \rangle}$. According to Lemma 3.3, the vector $v_{\ell, \ell'}$ in (3.34) is the unique singular vector of $\mathcal{H}_{\langle \ell, \ell' \rangle}$. Since $gl(n|m)_+$ in (2.51) is locally nilpotent by (1.18)-(1.22), V contains a singular vector. So $v_{\ell, \ell'} \in V$. By Lemma 3.4, $V = \mathcal{H}_{\langle \ell, \ell' \rangle}$, that is, $\mathcal{H}_{\langle \ell, \ell' \rangle}$ is irreducible. The necessity also follows from Lemma 3.3.

Assume $\ell + \ell' \leq n_1 + m + 1 - n_2$. Since Δ is locally nilpotent by (3.1) and (3.3), for any $0 \neq u \in \mathcal{A}_{\langle \ell, \ell' \rangle}$, there exists an element $\kappa(u) \in \mathbb{N}$ such that

$$\Delta^{\kappa(u)}(u) \neq 0 \quad \text{and} \quad \Delta^{\kappa(u)+1}(u) = 0. \quad (3.52)$$

Set

$$\Psi = \begin{cases} \sum_{i=0}^{\infty} \eta^i(\mathcal{H}_{\langle \ell-i, \ell'-i \rangle}) & \text{if } n_2 < n, \\ \sum_{i=0}^{\ell'} \eta^i(\mathcal{H}_{\langle \ell-i, \ell'-i \rangle}) & \text{if } n_2 = n. \end{cases} \quad (3.53)$$

Given $0 \neq u \in \mathcal{A}_{\langle \ell, \ell' \rangle}$, $\kappa(u) = 1$ implies $u \in \mathcal{H}_{\langle \ell, \ell' \rangle} \subset \Psi$. Suppose that $u \in \Psi$ whenever $\kappa(u) < r$ for some positive integer r . Assume $\kappa(u) = r$. First

$$v = \Delta^r(u) \in \mathcal{H}_{\langle \ell-r, \ell'-r \rangle} \subset \Psi. \quad (3.54)$$

Note

$$\Delta^r[\eta^r(v)] = r! \left[\prod_{i=1}^r (n_2 - n_1 - m + \ell + \ell' - r - i) \right] v \quad (3.55)$$

by (3.22). Thus we have either

$$u = \frac{1}{r! \left[\prod_{i=1}^r (n_2 - n_1 - m + \ell + \ell' - r - i) \right]} \eta^r(v) \in \Psi \quad (3.56)$$

or

$$\kappa \left(u - \frac{1}{r! \left[\prod_{i=1}^r (n_2 - n_1 - m + \ell + \ell' - r - i) \right]} \eta^r(v) \right) < r. \quad (3.57)$$

By induction,

$$u - \frac{1}{r! \left[\prod_{i=1}^r (n_2 - n_1 - m + \ell + \ell' - r - i) \right]} \eta^r(v) \in \Psi, \quad (3.58)$$

which implies $u \in \Psi$. Therefore, we have $\Psi = \mathcal{A}_{\langle \ell, \ell' \rangle}$. Since all $\eta^i(\mathcal{H}_{\langle \ell-i, \ell'-i \rangle})$ have distinct highest weights, the sums in (3.53) are direct sums.

This completes the proof of Theorem 2. \square

Remark 3.6. If $\ell + \ell' > n_1 + m + 1 - n_2$ and $\ell \leq n_1 + m + 1 - n_2$ when $n_2 = n$, the $gl(n|m)$ -module $\mathcal{H}_{\langle \ell, \ell' \rangle}$ is indecomposable. When $\ell + \ell' > n_1 + m + 1 - n_2$, $\mathcal{A}_{\langle \ell, \ell' \rangle}$ is not completely reducible.

Recall the notations in (2.104) and (2.105). The set

$$\begin{aligned} & \left\{ \sum_{i=0}^{\infty} \frac{(-1)^i x_{n_1+1}^i y_{n_1+1}^i}{\prod_{r=1}^i (\alpha_{n_1+1} + r)(\beta_{n_1+1} + r)} (\Delta - \partial_{x_{n_1+1}} \partial_{y_{n_1+1}})^i (x^\alpha y^\beta \theta_{\vec{j}} \vartheta_{\vec{k}}) \right. \\ & \quad \left| \alpha, \beta \in \mathbb{N}^n; \vec{j} \in \Gamma_{\ell_1}; \vec{k} \in \Gamma_{\ell_2}; \alpha_{n_1+1} \beta_{n_1+1} = 0; \ell_1, \ell_2 \in \overline{0, m}; \right. \\ & \quad \left. \sum_{r=n_1+1}^n \alpha_r - \sum_{i=1}^{n_1} \alpha_i + \ell_1 = \ell; \sum_{i=1}^{n_2} \beta_i - \sum_{r=n_2+1}^n \beta_r + \ell_2 = \ell' \right\} \end{aligned} \quad (3.59)$$

forms a basis of $\mathcal{H}_{\langle \ell, \ell' \rangle}$ by Lemma 2.6.

4 Proof of Theorem 3

In this section, we want to prove Theorem 3.

Set

$$\mathcal{K}_0 = \sum_{i,j=1}^n \mathbb{C}(E_{i,j} - E_{n+j,n+i}) + \sum_{r,s=1}^m \mathbb{C}(E_{2n+r,2n+s} - E_{2n+m+s,2n+m+r}), \quad (4.1)$$

$$\mathcal{K}_1 = \sum_{i=1}^n \sum_{r=1}^m [\mathbb{C}(E_{i,2n+r} - E_{2n+m+r,n+i}) + \mathbb{C}(E_{2n+r,i} + E_{n+i,2n+m+r})]. \quad (4.2)$$

Then $\mathcal{K} = \mathcal{K}_0 + \mathcal{K}_1$ forms a Lie sub-superalgebra of $gl(2n|2m)$ isomorphic to $gl(n|m)$. Let

$$\begin{aligned} osp(2n|2m)_0 &= \mathcal{K}_0 + \sum_{1 \leq i < j \leq n} [\mathbb{C}(E_{i,n+j} - E_{j,n+i}) + \mathbb{C}(E_{n+i,j} - E_{n+j,i})] \\ &\quad + \sum_{1 \leq r \leq s \leq m} [\mathbb{C}(E_{2n+r,2n+m+s} + E_{2n+s,2n+m+r}) \\ &\quad + \mathbb{C}(E_{2n+m+r,2n+s} + E_{2n+m+s,2n+r})], \end{aligned} \quad (4.3)$$

$$osp(2n|2m)_1 = \mathcal{K}_1 + \sum_{i=1}^n \sum_{r=1}^m [\mathbb{C}(E_{i,2n+m+r} + E_{2n+r,n+i}) + \mathbb{C}(E_{n+i,2n+r} - E_{2n+m+r,i})]. \quad (4.4)$$

The space $osp(2n|2m) = osp(2n|2m)_0 + osp(2n|2m)_1$ forms a simple Lie sub-superalgebra of $gl(2n|2m)$. Moreover, its Lie subalgebra

$$osp(2n|2m)_0 \cong o(2n, \mathbb{C}) \oplus sp(2n, \mathbb{C}). \quad (4.5)$$

Take settings in (1.8)-(1.11). Define a representation of $osp(2n|2m)$ on \mathcal{A} determined by

$$(E_{i,j} - E_{n+j,n+i})|_{\mathcal{A}} = x_i \partial_{x_j} - y_j \partial_{y_i}, \quad (E_{2n+r,2n+s} - E_{2n+m+s,2n+m+r})|_{\mathcal{A}} = \theta_r \partial_{\theta_s} - \vartheta_s \partial_{\vartheta_r}, \quad (4.6)$$

$$(E_{i,2n+r} - E_{2n+m+r,n+i})|_{\mathcal{A}} = x_i \partial_{\theta_r} - \vartheta_r \partial_{y_i}, \quad (E_{2n+r,i} + E_{n+i,2n+m+r})|_{\mathcal{A}} = \theta_r \partial_{x_i} + y_i \partial_{\vartheta_r}, \quad (4.7)$$

$$(E_{i,n+j} - E_{j,n+i})|_{\mathcal{A}} = x_i \partial_{y_j} - x_j \partial_{y_i}, \quad (E_{2n+m+r,2n+s} + E_{2n+m+s,2n+r})|_{\mathcal{A}} = \vartheta_r \partial_{\theta_s} + \vartheta_s \partial_{\theta_r}, \quad (4.8)$$

$$(E_{n+i,j} - E_{n+j,i})|_{\mathcal{A}} = y_i \partial_{x_j} - y_j \partial_{x_i}, \quad (E_{2n+r,2n+m+s} + E_{2n+s,2n+m+r})|_{\mathcal{A}} = \theta_r \partial_{\vartheta_s} + \theta_s \partial_{\vartheta_r}, \quad (4.9)$$

$$(E_{i,2n+m+r} + E_{2n+r,n+i})|_{\mathcal{A}} = x_i \partial_{\vartheta_r} + \theta_r \partial_{y_i}, \quad (E_{n+i,2n+r} - E_{2n+m+r,i})|_{\mathcal{A}} = y_i \partial_{\theta_r} - \vartheta_r \partial_{x_i} \quad (4.10)$$

for $i, j \in \overline{1, n}$ and $r, s \in \overline{1, m}$.

Recall that we write $\Theta_1 = \sum_{r=1}^m \mathbb{C}\theta_r$ and $\Theta_2 = \sum_{s=1}^m \mathbb{C}\vartheta_s$. For $k \in \mathbb{N}$, we denote

$$\mathcal{A}_k = \text{Span}\{x^\alpha y^\alpha \Theta_1^{\ell'_1} \Theta_2^{\ell'_2} \mid \alpha, \beta \in \mathbb{N}^n; \ell'_1, \ell'_2 \in \mathbb{N}; |\alpha| + \ell'_1 + |\beta| + \ell'_2 = k\}. \quad (4.11)$$

Again we take

$$\Delta = \sum_{i=1}^n \partial_{x_i} \partial_{y_i} + \sum_{r=1}^m \partial_{\theta_r} \partial_{\vartheta_r}, \quad \eta = \sum_{i=1}^n x_i y_i + \sum_{r=1}^m \theta_r \vartheta_r. \quad (4.12)$$

Set

$$\mathcal{W} = \left[\sum_{i=1}^n (\mathbb{C}x_i + \mathbb{C}y_i) + \sum_{r=1}^m (\mathbb{C}\theta_r + \mathbb{C}\vartheta_r) \right] \left[\sum_{j=1}^n (\mathbb{C}\partial_{x_j} + \mathbb{C}\partial_{y_j}) + \sum_{s=1}^m (\mathbb{C}\partial_{\theta_s} + \mathbb{C}\partial_{\vartheta_s}) \right]. \quad (4.13)$$

Then

$$osp(2n|2m)|_{\mathcal{A}} = \{T \in \mathcal{W} \mid T(\eta) = 0\}. \quad (4.14)$$

Moreover,

$$\xi \Delta = \Delta \xi, \quad \xi \eta = \eta \xi \quad \text{for } \xi \in osp(2n|2m) \quad (4.15)$$

as operators on \mathcal{A} . For $k \in \mathbb{N}$, the subspace

$$\mathcal{H}_k = \{f \in \mathcal{A}_k \mid \Delta(f) = 0\} \quad (4.16)$$

forms an $osp(2n|2m)$ -submodule. First we prove the first conclusion in Theorem 3:

Theorem 4.1. *Suppose $n > 1$. For $k \in \mathbb{N}$, \mathcal{H}_k is an irreducible $osp(2n|2m)$ -module if and only if $k \leq m+1-n$ or $k > 2(m+1-n)$. When $k \leq m+1-n$, $\mathcal{A}_k = \bigoplus_{i=0}^{\lfloor k/2 \rfloor} \eta^i \mathcal{H}_{k-2i}$ is a decomposition of irreducible $osp(2n|2m)$ -submodules.*

Proof. We take the subspace of diagonal matrices in $osp(2n|2m)$ as a Cartan subalgebras and the subspace

$$\begin{aligned}
osp(2n|2m)_+ = & \sum_{1 \leq i < j \leq n} [\mathbb{C}(E_{i,j} - E_{n+j,n+i}) + \mathbb{C}(E_{i,n+j} - E_{j,n+i})] \\
& + \sum_{1 \leq r < s \leq m} \mathbb{C}(E_{2n+r,2n+s} - E_{2n+m+s,2n+m+r}) + \sum_{1 \leq r \leq s \leq m} \mathbb{C}(E_{2n+r,2n+m+s} + E_{2n+s,2n+m+r}) \\
& + \sum_{i=1}^n \sum_{r=1}^m [\mathbb{C}(E_{i,2n+r} - E_{2n+m+r,n+i}) + \mathbb{C}(E_{i,2n+m+r} + E_{2n+r,n+i})]
\end{aligned} \tag{4.17}$$

as the space spanned by positive root vectors. An $osp(2n|2m)$ -singular vector v is a nonzero weight vector of $osp(2n|2m)$ such that $osp(2n|2m)_+(v) = 0$. We count singular vector up to a nonzero scalar multiple.

Observe $\mathcal{K}|_{\mathcal{A}} = gl(n|m)|_{\mathcal{A}}$. According to the arguments in (2.51)-(2.60), the homogeneous singular vectors of \mathcal{K} are:

$$\{\eta^{\ell_3} x_1^{\ell_1} y_n^{\ell_2} \vec{\vartheta}_s \mid \ell_i \in \mathbb{N}; s \in \overline{1, m+1}; \ell_2(s-1) = 0\}. \tag{4.18}$$

By (4.10), $(E_{n,2n+m+s} + E_{2n+s,2n})|_{\mathcal{A}} = x_n \partial_{\vartheta_s} + \theta_s \partial_{y_n}$. Thus the homogeneous singular vectors of $osp(2n|2m)$ are $\{\eta^{\ell_2} x_1^{\ell_1} \mid \ell_1, \ell_2 \in \mathbb{N}\}$. Moreover, (2.27) and (2.36) imply

$$\Delta(\eta^{\ell_2} x_1^{\ell_1}) = \ell_2(n - m + \ell_1 + \ell_2 - 1) = 0 \implies \ell_1 + n \leq m \text{ and } \ell_2 = m + 1 - n - \ell_1. \tag{4.19}$$

In this case, $\eta^{m+1-n-\ell_1} x_1^{\ell_1} \in \mathcal{H}_{2(m+1-n)-\ell_1}$. Thus

$$\mathcal{H}_k \text{ has a unique singular vector if and only if } k \leq m+1-n \text{ or } k > 2(m+1-n), \tag{4.20}$$

and

$$\mathcal{H}_k \text{ has two singular vectors when } m+1-n < k \leq 2(m+1-n). \tag{4.21}$$

Note $x_1^k \in \mathcal{H}_k$. Let U be the $osp(2n|2m)$ -submodule generated by x_1^k . Repeatedly applying $(E_{n+1,2n+r} - E_{2n+m+r,1})|_{\mathcal{A}} = y_1 \partial_{\theta_r} - \vartheta_r \partial_{x_1}$ (cf. (4.10)), we get

$$x_1^{\ell} \vec{\vartheta}_s \in U \quad \text{for } \ell + m + 1 - s = k. \tag{4.22}$$

According to (4.8), $(E_{n+1,n} - E_{2n,1})|_{\mathcal{A}} = y_1 \partial_{x_n} - y_n \partial_{x_1}$. Thus

$$x_1^{\ell_1} y_n^{\ell_2} \vec{\vartheta}_1 \in U \quad \text{for } \ell_1 + \ell_2 + m = k \tag{4.23}$$

when $k \geq m$. Since

$$\mathcal{H}_k = \sum_{i=0}^k \mathcal{H}_{i,k-i}, \tag{4.24}$$

Lemma 2.5, (4.22) and (4.23) imply $U = \mathcal{H}_k$. So \mathcal{H}_k is an $osp(2n|2m)$ -module generated by x_1^k .

Suppose $k \leq m+1-n$ or $k > 2(m+1-n)$. Let M be a nonzero $osp(2n|2m)$ -submodule of \mathcal{H}_k . By (4.20), \mathcal{H}_k contains a unique singular vector x_1^k . Thus $x_1^k \in M$. By the above paragraph, $\mathcal{H}_k \subset M$. Hence \mathcal{H}_k is irreducible.

If \mathcal{H}_k is irreducible, (4.21) implies $k \leq m+1-n$ or $k > 2(m+1-n)$.

Assume $k \leq m + 1 - n$. Note

$$\mathcal{A}_j = \sum_{\ell=0}^j \mathcal{A}_{\ell, j-\ell} \quad \text{for } j \in \mathbb{N}. \quad (4.25)$$

By Theorem 1,

$$\mathcal{A}_k = \bigoplus_{\ell=0}^k \mathcal{A}_{\ell, k-\ell} = \bigoplus_{\ell=0}^k \bigoplus_{i=0}^{\lfloor k/2 \rfloor} \eta^i \mathcal{H}_{\ell-i, k-\ell-i} = \bigoplus_{i=0}^{\lfloor k/2 \rfloor} \eta^i \mathcal{H}_{k-2i}. \quad \square \quad (4.26)$$

Fix $n_1, n_2 \in \overline{1, n}$ with $n_1 + 1 < n_2$. Changing operators $\partial_{x_r} \mapsto -x_r$, $x_r \mapsto \partial_{x_r}$ for $r \in \overline{1, n_1}$ and $\partial_{y_s} \mapsto -y_s$, $y_s \mapsto \partial_{y_s}$ for $s \in \overline{n_2 + 1, n}$ in (4.6)-(4.10), we get a new representation of $osp(2n|2m)$ on \mathcal{A} determined by

$$E_{2n+r, 2n+s}|_{\mathcal{A}} = \theta_r \partial_{\theta_s}, \quad E_{2n+m+r, 2n+m+s}|_{\mathcal{A}} = \vartheta_r \partial_{\vartheta_s}, \quad (4.27)$$

$$E_{2n+r, 2n+m+s}|_{\mathcal{A}} = \theta_r \partial_{\vartheta_s}, \quad E_{2n+m+r, 2n+s}|_{\mathcal{A}} = \vartheta_r \partial_{\theta_s}, \quad (4.28)$$

$$E_{i,j}|_{\mathcal{A}} = \begin{cases} -x_j \partial_{x_i} - \delta_{i,j} & \text{if } i, j \in \overline{1, n_1}; \\ \partial_{x_i} \partial_{x_j} & \text{if } i \in \overline{1, n_1}, j \in \overline{n_1 + 1, n}; \\ -x_i x_j & \text{if } i \in \overline{n_1 + 1, n}, j \in \overline{1, n_1}; \\ x_i \partial_{x_j} & \text{if } i, j \in \overline{n_1 + 1, n}; \end{cases} \quad (4.29)$$

$$E_{n+i, n+j}|_{\mathcal{A}} = \begin{cases} y_i \partial_{y_j} & \text{if } i, j \in \overline{1, n_2}; \\ -y_i y_j & \text{if } i \in \overline{1, n_2}, j \in \overline{n_2 + 1, n}; \\ \partial_{y_i} \partial_{y_j} & \text{if } i \in \overline{n_2 + 1, n}, j \in \overline{1, n_2}; \\ -y_j \partial_{y_i} - \delta_{i,j} & \text{if } i, j \in \overline{n_2 + 1, n}; \end{cases} \quad (4.30)$$

$$E_{i, n+j}|_{\mathcal{A}} = \begin{cases} \partial_{x_i} \partial_{y_j} & \text{if } i \in \overline{1, n_1}, j \in \overline{1, n_2}; \\ -y_j \partial_{x_i} & \text{if } i \in \overline{1, n_1}, j \in \overline{n_2 + 1, n}; \\ x_i \partial_{y_j} & \text{if } i \in \overline{n_1 + 1, n}, j \in \overline{1, n_2}; \\ -x_i y_j & \text{if } i \in \overline{n_1 + 1, n}, j \in \overline{n_2 + 1, n}; \end{cases} \quad (4.31)$$

$$E_{n+i, j}|_{\mathcal{A}} = \begin{cases} -x_j y_i & \text{if } j \in \overline{1, n_1}, i \in \overline{1, n_2}; \\ -x_j \partial_{y_i} & \text{if } j \in \overline{1, n_1}, i \in \overline{n_2 + 1, n}; \\ y_i \partial_{x_j} & \text{if } j \in \overline{n_1 + 1, n}, i \in \overline{1, n_2}; \\ \partial_{x_j} \partial_{y_i} & \text{if } j \in \overline{n_1 + 1, n}, i \in \overline{n_2 + 1, n}; \end{cases} \quad (4.32)$$

$$E_{i, 2n+r}|_{\mathcal{A}} = \begin{cases} \partial_{x_i} \partial_{\theta_r} & \text{if } i \in \overline{1, n_1}; \\ x_i \partial_{\theta_r} & \text{if } i \in \overline{n_1 + 1, n}; \end{cases} \quad E_{i, 2n+m+r}|_{\mathcal{A}} = \begin{cases} \partial_{x_i} \partial_{\vartheta_r} & \text{if } i \in \overline{1, n_1}; \\ x_i \partial_{\vartheta_r} & \text{if } i \in \overline{n_1 + 1, n}; \end{cases} \quad (4.33)$$

$$E_{2n+r, i}|_{\mathcal{A}} = \begin{cases} -x_i \theta_r & \text{if } i \in \overline{1, n_1}; \\ \theta_r \partial_{x_i} & \text{if } i \in \overline{n_1 + 1, n}; \end{cases} \quad E_{2n+m+r, i}|_{\mathcal{A}} = \begin{cases} -x_i \vartheta_r & \text{if } i \in \overline{1, n_1}; \\ \vartheta_r \partial_{x_i} & \text{if } i \in \overline{n_1 + 1, n}; \end{cases} \quad (4.34)$$

$$E_{n+i, 2n+r}|_{\mathcal{A}} = \begin{cases} y_i \partial_{\theta_r} & \text{if } i \in \overline{1, n_2}; \\ \partial_{y_i} \partial_{\theta_r} & \text{if } i \in \overline{n_2 + 1, n}; \end{cases} \quad (4.35)$$

$$E_{n+i, 2n+m+r}|_{\mathcal{A}} = \begin{cases} y_i \partial_{\vartheta_r} & \text{if } i \in \overline{1, n_2}; \\ \partial_{y_i} \partial_{\vartheta_r} & \text{if } i \in \overline{n_2 + 1, n}; \end{cases} \quad (4.36)$$

$$E_{2n+r, n+i}|_{\mathcal{A}} = \begin{cases} \theta_r \partial_{y_i} & \text{if } i \in \overline{1, n_2}; \\ -y_i \theta_r & \text{if } i \in \overline{n_2 + 1, n}; \end{cases} \quad (4.37)$$

$$E_{2n+m+r, n+i}|_{\mathcal{A}} = \begin{cases} \vartheta_r \partial_{y_i} & \text{if } i \in \overline{1, n_2}; \\ -y_i \vartheta_r & \text{if } i \in \overline{n_2 + 1, n}; \end{cases} \quad (4.38)$$

for $i, j \in \overline{1, n}$ and $r, s \in \overline{1, m}$.

The related Laplace operator becomes

$$\Delta = - \sum_{i=1}^{n_1} x_i \partial_{y_i} + \sum_{r=n_1+1}^{n_2} \partial_{x_r} \partial_{y_r} - \sum_{s=n_2+1}^n y_s \partial_{x_s} + \sum_{r=1}^m \partial_{\theta_r} \partial_{\vartheta_r} \quad (4.39)$$

and its dual

$$\eta = \sum_{i=1}^{n_1} y_i \partial_{x_i} + \sum_{r=n_1+1}^{n_2} x_r y_r + \sum_{s=n_2+1}^n x_s \partial_{y_s} + \sum_{r=1}^m \theta_r \vartheta_r. \quad (4.40)$$

It can be verified that (4.15) holds again. Denote

$$\begin{aligned} \mathcal{A}_{\langle k \rangle} = & \text{Span}\{x^\alpha y^\beta \Theta_1^{\ell'_1} \Theta_2^{\ell'_2} \mid \alpha, \beta \in \mathbb{N}^n; \ell'_1, \ell'_2 \in \mathbb{N}; \\ & \sum_{r=n_1+1}^n \alpha_r - \sum_{i=1}^{n_1} \alpha_i + \sum_{i=1}^{n_2} \beta_i - \sum_{r=n_2+1}^n \beta_r + \ell'_1 + \ell'_2 = \ell_2\} \end{aligned} \quad (4.41)$$

for $k \in \mathbb{Z}$.

Again we set $\mathcal{H}_{\langle k \rangle} = \{f \in \mathcal{A}_{\langle k \rangle} \mid \Delta(f) = 0\}$. Next we prove the second conclusion in Theorem 3:

Theorem 4.2. *Let $k \in \mathbb{Z}$. The $osp(2n|2m)$ -module $\mathcal{H}_{\langle k \rangle}$ is irreducible if and only if $k \leq n_1 + m + 1 - n_2$. When $k \leq n_1 + m + 1 - n_2$, $\mathcal{A}_{\langle k \rangle} = \bigoplus_{i=0}^{\infty} \eta^i(\mathcal{H}_{\langle k-2i \rangle})$ is the decomposition of irreducible $osp(2n|2m)$ -submodules.*

Proof. Observe $\mathcal{K}|_{\mathcal{A}} = gl(n|m)|_{\mathcal{A}}$ in terms the representation of $gl(n|m)$ given in (1.18)-(1.22). Lemma 3.2 says that the homogeneous singular vectors of \mathcal{K} are of the form:

$$\eta^{\ell_1}(x_i^{m_1} y_j^{m_2} \vec{\vartheta}_s) \text{ with } \ell_1, m_1, m_2 \in \mathbb{N}, s \in \overline{1, m+1} \quad (4.42)$$

and

$$(i, j) \in \{(n_1, n_2), (n_1, n_2 + 1 - \delta_{n_2, n}), (n_1 + 1, n_2), (n_1 + 1, n_2 + 1 - \delta_{n_2, n})\}. \quad (4.43)$$

Claim 1. For $k \in \mathbb{N}$, $\mathcal{H}_{\langle k \rangle}$ is an $osp(2n|2m)$ -module generated by $x_{n_1+1}^k$ and $\mathcal{H}_{\langle -k \rangle}$ is an $osp(2n|2m)$ -module generated by $x_{n_1}^k$.

Let V be the $osp(2n|2m)$ -module generated by $x_{n_1+1}^k \in \mathcal{H}_{\langle k \rangle}$. By (4.31),

$$(E_{n_2+1, n_1+1} - E_{n_1+1, n_2+1})|_{\mathcal{A}} = x_{n_2+1} \partial_{y_{n_1+1}} + x_{n_1+1} y_{n_2+1}. \quad (4.44)$$

Thus

$$(E_{n_2+1, n_1+1} - E_{n_1+1, n_2+1})^{k_1}(x_{n_1+1}^k) = x_{n_1+1}^{k+k_1} y_{n_2+1}^{k_1} \in V. \quad (4.45)$$

According to (4.32),

$$(E_{n+n_2, n_1+1} - E_{n+n_1+1, n_2})|_{\mathcal{A}} = y_{n_2} \partial_{x_{n_1+1}} - y_{n_1+1} \partial_{x_{n_2}}. \quad (4.46)$$

Repeatedly applying (4.46) to $x_{n_1+1}^k$, we obtain

$$x_{n_1+1}^{k_1} y_{n_2}^{k_2} \in V \quad \text{for } k_1, k_2 \in \mathbb{N} \text{ such that } k_1 + k_2 = k. \quad (4.47)$$

Note that (4.34) and (4.35) imply

$$(E_{n+n_1+1, 2n+r} - E_{2n+m+r, n_1+1})|_{\mathcal{A}} = y_{n_1+1} \partial_{\theta_r} - \vartheta_r \partial_{x_{n_1+1}} \quad (4.48)$$

Applying (4.48) with various r to (4.45) and (4.47), we obtain

$$x_{n_1+1}^{\ell_1} y_j^{\ell_2} \vec{\vartheta}_s \in \mathcal{H}_{\langle k \rangle} \text{ with } j \in \{n_2, n_2 + 1\} \implies x_{n_1+1}^{\ell_1} y_j^{\ell_2} \vec{\vartheta}_s \in V. \quad (4.49)$$

In terms of (3.34),

$$v_{\ell, \ell'} \in V, \quad \ell + \ell' = k. \quad (4.50)$$

By Lemma 3.4,

$$\mathcal{H}_{\langle \ell, \ell' \rangle} \subset V, \quad \ell + \ell' = k. \quad (4.51)$$

Therefore

$$\mathcal{H}_{\langle k \rangle} = \bigoplus_{\ell, \ell' \in \mathbb{Z}; \ell + \ell' = k} \mathcal{H}_{\langle \ell, \ell' \rangle} \subset V. \quad (4.52)$$

Suppose $k > 0$. Let U be the $osp(2n|2m)$ -module generated by $x_{n_1}^k \mathcal{H}_{\langle -k \rangle}$. Observe

$$(E_{n_2+1, n+n_1} - E_{n_1, n+n_2})|_{\mathcal{A}} = x_{n_2+1} \partial_{y_{n_1}} + y_{n_2+1} \partial_{x_{n_1}} \quad (4.53)$$

by (4.31), and

$$(E_{n+n_1, n_2} - E_{n+n_2, n_1})|_{\mathcal{A}} = y_{n_1} \partial_{x_{n_1}} + x_{n_1} y_{n_1} \quad (4.54)$$

by (4.32). Repeatedly applying the above two equations to $x_{n_1}^k$, we have

$$x_{n_1}^{k_1} y_{n_2+1}^{k_2}, x_{n_1}^{k+k_3} y_{n_2}^{k_3} \in U \quad \text{for } k_1, k_2, k_3 \in \mathbb{N} \text{ such that } k_1 + k_2 = k. \quad (4.55)$$

According to (4.34) and (4.35),

$$(E_{n+n_1, 2n+r} - E_{2n+m+r, n_1})|_{\mathcal{A}} = y_{n_1} \partial_{\theta_r} + x_{n_1} \vartheta_r. \quad (4.56)$$

Applying (4.56) with various r to (4.55), we find

$$x_{n_1}^{\ell_1} y_j^{\ell_2} \vec{\vartheta}_s \in \mathcal{H}_{\langle -k \rangle} \text{ with } j \in \{n_2, n_2 + 1\} \implies x_{n_1}^{\ell_1} y_j^{\ell_2} \vec{\vartheta}_s \in V. \quad (4.57)$$

In terms of (3.34),

$$v_{\ell, \ell'} \in V, \quad \ell + \ell' = -k. \quad (4.58)$$

Lemma 3.4 gives

$$\mathcal{H}_{\langle \ell, \ell' \rangle} \subset V, \quad \ell + \ell' = -k. \quad (4.59)$$

Thus

$$\mathcal{H}_{\langle -k \rangle} = \bigoplus_{\ell, \ell' \in \mathbb{Z}; \ell + \ell' = -k} \mathcal{H}_{\langle \ell, \ell' \rangle} \subset V. \quad (4.60)$$

This prove Claim 1.

Claim 2. For $n_1 + m + 1 - n_2 \geq k \in \mathbb{Z}$, any nonzero $osp(2n|2m)$ -submodule of $\mathcal{H}_{\langle k \rangle}$ contains $x_{n_1+1}^k$ if $k \geq 0$ or $x_{n_1}^{-k}$ when $k < 0$.

Note

$$x_{n_1}^{m_1} y_{n_2}^{m_2} \vec{\vartheta}_s \in \mathcal{H}_{\langle m+m_2+1-s-m_1 \rangle}, \quad x_{n_1}^{m_1} y_{n_2+1}^{m_2} \vec{\vartheta}_s \in \mathcal{H}_{\langle m+1-s-m_1-m_2 \rangle}, \quad (4.61)$$

$$x_{n_1+1}^{m_1} y_{n_2}^{m_2} \vec{\vartheta}_s \in \mathcal{H}_{\langle m+m_1+m_2+1-s \rangle}, \quad x_{n_1+1}^{m_1} y_{n_2+1}^{m_2} \vec{\vartheta}_s \in \mathcal{H}_{\langle m+m_1+1-s-m_2 \rangle}. \quad (4.62)$$

For $\ell_1 \in \mathbb{N} + 1$ and $f \in \mathcal{H}_{\langle k' \rangle}$ with $k' \in \mathbb{Z}$, (2.27) and (3.21) imply

$$\Delta \eta^{\ell_1}(f) = \ell_1(n_2 - n_1 - m + k' + \ell_1 - 1)\eta^{\ell_1-1}(f). \quad (4.63)$$

Thus

$$\Delta \eta^{\ell_1}(f) = 0 \iff k' \leq n_1 + m - n_2 \text{ and } \ell_1 = n_1 + m - n_2 - k' + 1. \quad (4.64)$$

If the condition holds, then

$$\eta^{\ell_1}(f) \in \mathcal{H}_{\langle 2(n_1+m+1-n_2)-k' \rangle}. \quad (4.65)$$

Moreover,

$$2(n_1 + m + 1 - n_2) - k' = n_1 + m + 2 - n_2 + (n_1 + m - n_2 - k') \geq n_1 + m + 2 - n_2. \quad (4.66)$$

This shows that

$$\mathcal{H}_{\langle k_1 \rangle} \cap \left(\bigcup_{i=1}^{\infty} \eta^i(\mathcal{H}) \right) \neq \{0\} \iff k_1 \geq n_1 + m + 2 - n_2. \quad (4.67)$$

Suppose $k \leq n_1 + m + 1 - n_2$. Then the singular vectors of \mathcal{K} in $\mathcal{H}_{\langle k \rangle}$ are of the form

$$x_i^{m_1} y_j^{m_2} \vec{\partial}_s \text{ with } m_1, m_2 \in \mathbb{N}, s \in \overline{1, m+1} \quad (4.68)$$

with (4.43). Observe that

$$(E_{n_1+1, n+n_2} - E_{n_2, n+n_1+1})|_{\mathcal{A}} = x_{n_1+1} \partial_{y_{n_2}} - x_{n_2} \partial_{y_{n_1+1}} \quad (4.69)$$

by (4.29), and

$$(E_{n+n_2+1, n_1} - E_{n+n_1, n_2+1})|_{\mathcal{A}} = -x_{n_1} \partial_{y_{n_2+1}} - y_{n_1} \partial_{x_{n_2+1}} \quad (4.70)$$

by (4.32). According to (4.33) and (4.37),

$$(E_{n_1+1, 2n+m+r} + E_{2n+r, n+n_1+1})|_{\mathcal{A}} = x_{n_1+1} \partial_{\vartheta_r} + \theta_r \partial_{y_{n_1+1}}. \quad (4.71)$$

Let M be any nonzero $osp(2n|2m)$ -submodule of $\mathcal{H}_{\langle k \rangle}$. Then M contains at least one of the \mathcal{K} -singular vectors in (4.68). Applying (4.68)-(4.71), we get

$$x_{n_1}^{k_1} x_{n_2}^{k_2} \in M \text{ for some } k_1, k_2 \in \mathbb{N} \text{ such that } k_2 - k_1 = k. \quad (4.72)$$

By (4.29) and (4.30),

$$(E_{n_1, n_1+1} - E_{n+n_1+1, n+n_1})|_{\mathcal{A}} = \partial_{x_{n_1}} \partial_{x_{n_1+1}} - y_{n_1+1} \partial_{y_{n_1}}. \quad (4.73)$$

Repeatedly applying (4.73) to (4.72), we obtain $x_{n_1+1}^k \in M$ if $k \geq 0$ or $x_{n_1}^{-k} \in M$ when $k < 0$.

The above claims show that $\mathcal{H}_{\langle k \rangle}$ is an irreducible $osp(2n|2m)$ -module if $k \leq n_1 + m + 1 - n_2$.

Note

$$\mathcal{A}_{\langle j \rangle} = \sum_{\ell \in \mathbb{Z}} \mathcal{A}_{\langle \ell, j-\ell \rangle}, \quad \mathcal{H}_{\langle j \rangle} = \sum_{\ell \in \mathbb{Z}} \mathcal{H}_{\langle \ell, j-\ell \rangle} \quad \text{for } j \in \mathbb{Z}. \quad (4.74)$$

By (4.67), $k \leq n_1 + m + 1 - n_2$ if $\mathcal{H}_{\langle k \rangle}$ is irreducible. When $k \leq n_1 + m + 1 - n_2$, Theorem 2 implies

$$\mathcal{A}_{\langle k \rangle} = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{A}_{\langle \ell, k-\ell \rangle} = \bigoplus_{\ell \in \mathbb{Z}} \bigoplus_{i=0}^{\infty} \eta^i \mathcal{H}_{\langle \ell-i, k-\ell-i \rangle} = \bigoplus_{i=0}^{\infty} \eta^i \mathcal{H}_{\langle k-2i \rangle}. \quad \square \quad (4.75)$$

We remark that $\mathcal{H}_{\langle k \rangle}$ is an indecomposable $osp(2n|2m)$ -module if $k \geq n_1 + m + 2 - n_2$ by Claim 1 and (4.67). This also implies that $\mathcal{A}_{\langle k \rangle}$ is not completely reducible $osp(2n|2m)$ -module when $k \geq n_1 + m + 2 - n_2$.

5 Proof of Theorem 4

In this section, we want to prove Theorem 3.

Note

$$osp(2n+1|2m)_0 = osp(2n|2m)_0 + \sum_{i=1}^n [\mathbb{C}(E_{0,i} - E_{n+i,0}) + \mathbb{C}(E_{0,n+i} - E_{i,0})], \quad (5.1)$$

$$\begin{aligned} osp(2n+1|2m)_1 &= osp(2n|2m)_1 + \sum_{r=1}^m [\mathbb{C}(E_{0,2n+r} - E_{2n+m+r,0}) \\ &\quad + \mathbb{C}(E_{0,2n+m+r} + E_{2n+r,0})]. \end{aligned} \quad (5.2)$$

The Lie superalgebra $osp(2n+1|2m) = osp(2n+1|2m)_0 + osp(2n+1|2m)_1$ is a Lie sub-superalgebra of $gl(2n+1|2m)$. Take settings in (1.28) and (1.29). Now $osp(2n|2m)$ acts on \mathcal{B} by the differential operators in (4.6)-(4.10), namely, we change the subindex $|_{\mathcal{A}}$ to $|_{\mathcal{B}}$. Extend the representation of $osp(2n|2m)$ on \mathcal{B} to a representation of $osp(2n+1|2m)$ on \mathcal{B} by:

$$(E_{0,i} - E_{n+i,0})|_{\mathcal{B}} = x_0 \partial_{x_i} - y_i \partial_{x_0}, \quad (E_{0,n+i} - E_{i,0})|_{\mathcal{B}} = x_0 \partial_{y_i} - x_i \partial_{x_0}, \quad (5.3)$$

$$(E_{0,2n+r} - E_{2n+m+r,0})|_{\mathcal{B}} = x_0 \partial_{\theta_r} - \vartheta_r \partial_{x_0}, \quad (E_{0,2n+m+r} + E_{2n+r,0})|_{\mathcal{B}} = x_0 \partial_{\vartheta_r} + \theta_r \partial_{x_0} \quad (5.4)$$

for $i \in \overline{1, n}$ and $r \in \overline{1, m}$.

Set

$$\mathcal{W}' = \left[\sum_{i=0}^n (\mathbb{C}x_i + \mathbb{C}y_i) + \sum_{r=1}^m (\mathbb{C}\theta_r + \mathbb{C}\vartheta_r) \right] \left[\sum_{j=0}^n (\mathbb{C}\partial_{x_j} + \mathbb{C}\partial_{y_j}) + \sum_{s=1}^m (\mathbb{C}\partial_{\theta_s} + \mathbb{C}\partial_{\vartheta_s}) \right]. \quad (5.5)$$

Then

$$osp(2n+1|2m)|_{\mathcal{B}} = \{T \in \mathcal{W}' \mid T(\eta') = 0\}. \quad (5.6)$$

Define

$$\mathcal{H}' = \{f \in \mathcal{B}' \mid \Delta'(f) = 0\}, \quad \mathcal{H}'_k = \mathcal{H}' \bigcap \mathcal{B}_k. \quad (5.7)$$

Again we take the subspace of diagonal matrices in $osp(2n+1|2m)$ as a Cartan sub-algebra and take the space spanned by positive roots:

$$osp(2n+1|2m)_+ = \mathbb{C}(E_{0,n+i} - E_{i,0}) + \mathbb{C}(E_{0,2n+m+r} + E_{2n+r,0}) + osp(2n|2m)_+. \quad (5.8)$$

An $osp(2n+1|2m)$ -singular vector v is a nonzero weight vector of $osp(2n+1|2m)$ such that $osp(2n+1|2m)_+(v) = 0$. We count singular vector up to a nonzero scalar multiple. According to the proof of Theorem 4.1, any singular vector of $osp(2n+1|2m)$ must be in $\mathbb{C}[x_0, x_1, \eta]$, where

$$\eta = \sum_{i=1}^n x_i y_i + \sum_{r=1}^m \theta_r \vartheta_r. \quad (5.9)$$

Note that $\eta' = x_0^2 + 2\eta$. By (5.8), x_1^k is a singular vector of $osp(2n+1|2m)$ for any $k \in \mathbb{N}$. Thus a homogeneous singular vector of $osp(2n+1|2m)$ must be of the form

$$f = \sum_{i=0}^{\ell} b_i x_0^{2i+\iota} \eta^{\ell-i} x_1^k, \quad (5.10)$$

where $b_i \in \mathbb{C}$, $\ell, k \in \mathbb{N}$ and $\iota = 0, 1$. Note

$$(E_{0,n+i} - E_{i,0})(f) = (x_0 \partial_{y_i} - x_i \partial_{x_0})(f) = 0 \iff f = b_0 (\eta')^{\ell} x_1^k. \quad (5.11)$$

Thus $\{(\eta')^{\ell} x_1^k \mid \ell, k \in \mathbb{N}\}$ are all the homogeneous singular vectors of $osp(2n+1|2m)$ in \mathcal{B} .

Observe that

$$[\Delta', \eta'] = 2 + 4(n-m) + 4[x_0 \partial_{x_0} + \sum_{i=1}^n (x_i \partial_{x_i} + y_i \partial_{y_i}) + \sum_{r=1}^m (\theta_r \partial_{\theta_r} + \vartheta_r \partial_{\vartheta_r})] \quad (5.12)$$

by (2.27) and (2.36). So

$$\Delta'((\eta')^{\ell} g) = 2\ell(1 + 2(n-m+k+\ell-1))(\eta')^{\ell-1} g \quad \text{for } g \in \mathcal{H}_k \quad (5.13)$$

Thus

$$\mathcal{H}'_k \text{ has a unique singular vector } x_1^k \text{ for any } k \in \mathbb{N}. \quad (5.14)$$

Indeed we have the first conclusion in Theorem 4:

Theorem 5.1. *For any $k \in \mathbb{N}$, \mathcal{H}'_k is an irreducible $osp(2n+1|2m)$ -module. Moreover, $\mathcal{B} = \bigoplus_{\ell,k=0}^{\infty} (\eta')^{\ell} \mathcal{H}_k$ is a direct sum of irreducible $osp(2n+1|2m)$ -submodules.*

Proof. By the arguments in (3.52)-(3.58), we only need to prove that \mathcal{H}'_k is an $osp(2n+1|2m)$ -module generated by x_1^k . For $\iota = 0, 1$, we define

$$T_{\iota} = \sum_{i=0}^{\infty} \frac{(-2)^i x_0^{2i+\iota}}{(2i+\iota)!} \Delta^i, \quad \Delta = \sum_{i=1}^n \partial_{x_i} \partial_{y_i} + \sum_{r=1}^m \partial_{\theta_r} \partial_{\vartheta_r}. \quad (5.15)$$

We take the notations in (2.104) and (2.105). By Lemma 2.6,

$$\begin{aligned} \mathcal{H}'_k &= \text{Span}\{T_{\iota}(x^{\alpha} y^{\beta} \theta_{\vec{j}} \vartheta_{\vec{j}'}) \mid \alpha, \beta \in \mathbb{N}^n; \vec{j} \in \Gamma_{k_1}, \vec{j}' \in \Gamma_{k_2}; \\ &\quad \iota \in \{0, 1\}; |\alpha| + |\beta| + k_1 + k_2 + \iota = k\}. \end{aligned} \quad (5.16)$$

Let U be the $osp(2n+1|2m)$ -module generated by $x_1^k \in \mathcal{H}_k$. Since $o(2n+1, \mathbb{C})$ is a subalgebra of $osp(2n+1|2m)$, the known results of the representation of $o(2n+1, \mathbb{C})$ on $\mathbb{C}[x_0, x_1, \dots, x_{2n}]$ show

$$T_{\iota}(x^{\alpha} y^{\beta}) \in U \quad \text{for } \alpha, \beta \in \mathbb{N}^n; |\alpha| + |\beta| = k. \quad (5.17)$$

Repeatedly applying $(E_{2n+r,i} + E_{n+i,2n+m+r})|_{\mathcal{A}} = \theta_r \partial_{x_i} + y_i \partial_{\vartheta_r}$ to (5.17) with $i \in \overline{1, n}$ and $r \in \overline{1, m}$, we obtain

$$T_l(x^\alpha y^\beta \theta_{\vec{j}}) \in U \quad \text{for } \alpha, \beta \in \mathbb{N}^n, \vec{j} \in \Gamma_{k_1}; |\alpha| + |\beta| + k_1 = k. \quad (5.18)$$

Finally, we get $U = \mathcal{H}'_k$ by repeatedly applying $(E_{i,2n+r} - E_{2n+m+r,n+i})|_{\mathcal{B}} = x_i \partial_{\theta_r} - \vartheta_r \partial_{y_i}$ to (5.18) with $i \in \overline{1, n}$ and $r \in \overline{1, m}$. \square

Next $osp(2n|2m)$ acts on \mathcal{B} via the differential operators in (4.27)-(4.38), namely, we change the subindex $|_{\mathcal{A}}$ to $|_{\mathcal{B}}$. Moreover, we extend the representation of $osp(2n|2m)$ on \mathcal{B} to a representation of $osp(2n+1|2m)$ on \mathcal{B} by:

$$(E_{0,i} - E_{n+i,0})|_{\mathcal{B}} = \begin{cases} -x_0 x_i - y_i \partial_{x_0} & \text{if } i \in \overline{1, n_1}, \\ x_0 \partial_{x_i} - y_i \partial_{x_0} & \text{if } i \in \overline{n_1+1, n_2}, \\ x_0 \partial_{x_i} - \partial_{x_0} \partial_{y_i} & \text{if } i \in \overline{n_2+1, n}; \end{cases} \quad (5.19)$$

$$(E_{0,n+i} - E_{n,0})|_{\mathcal{B}} = \begin{cases} x_0 \partial_{y_i} - \partial_{x_i} \partial_{x_0} & \text{if } i \in \overline{1, n_1}, \\ x_0 \partial_{y_i} - x_i \partial_{x_0} & \text{if } i \in \overline{n_1+1, n_2}, \\ -x_0 y_i - x_i \partial_{x_0} & \text{if } i \in \overline{n_2+1, n}; \end{cases} \quad (5.20)$$

$$(E_{0,2n+r} - E_{2n+m+r,0})|_{\mathcal{B}} = x_0 \partial_{\theta_r} - \vartheta_r \partial_{x_0}, \quad (E_{0,2n+m+r} + E_{2n+r,0})|_{\mathcal{B}} = x_0 \partial_{\vartheta_r} + \theta_r \partial_{x_0} \quad (5.21)$$

for $i \in \overline{1, n}$ and $r \in \overline{1, m}$.

Now the corresponding Laplace operator becomes

$$\Delta' = \partial_{x_0}^2 + 2\Delta, \quad \Delta = - \sum_{i=1}^{n_1} x_i \partial_{y_i} + \sum_{r=n_1+1}^{n_2} \partial_{x_r} \partial_{y_r} - \sum_{s=n_2+1}^n y_s \partial_{x_s} + \sum_{r=1}^m \partial_{\theta_r} \partial_{\vartheta_r} \quad (5.22)$$

and its dual

$$\eta' = x_0^2 + 2\eta, \quad \eta = \sum_{i=1}^{n_1} y_i \partial_{x_i} + \sum_{r=n_1+1}^{n_2} x_r y_r + \sum_{s=n_2+1}^n x_s \partial_{y_s} + \sum_{r=1}^m \theta_r \vartheta_r. \quad (5.23)$$

We take the notation in (4.41) and set

$$\mathcal{B}_{\langle k \rangle} = \sum_{i=0}^{\infty} \mathcal{A}_{\langle k-i \rangle} x_0^i, \quad \mathcal{H}'_{\langle k \rangle} = \{f \in \mathcal{B}_{\langle k \rangle} \mid \Delta'(f) = 0\}. \quad (5.24)$$

Then we have the second conclusion in Theorem 4:

Theorem 5.2. *For any $k \in \mathbb{Z}$, $\mathcal{H}'_{\langle k \rangle}$ is an irreducible $osp(2n+1|2m)$ -module. Moreover, $\mathcal{B} = \bigoplus_{\ell,k=0}^{\infty} (\eta')^\ell \mathcal{H}_{\langle k \rangle}$ is a direct sum of irreducible $osp(2n+1|2m)$ -submodules.*

Proof. We define T_l as in (5.15) with Δ in (5.22). By Lemma 2.6,

$$\mathcal{H}'_{\langle k \rangle} = T_0(\mathcal{A}_{\langle k \rangle}) + T_1(\mathcal{A}_{\langle k-1 \rangle}) \quad \text{for } k \in \mathbb{Z}. \quad (5.25)$$

Since $\Delta\xi = \xi\Delta$ for $\xi \in osp(2n|2m)$, we have

$$\xi(T_l(f)) = T_l(\xi(f)) \quad \text{for } \xi \in osp(2n|2m), f \in \mathcal{A}. \quad (5.26)$$

First we consider $\mathcal{H}'_{\langle k \rangle}$ with $k \in \mathbb{N}$. Let V be any nonzero $osp(2n+1|2m)$ -submodule of $\mathcal{H}'_{\langle k \rangle}$. According to the arguments in paragraph of (4.68)-(4.73), V contains some $T_l(\eta^\ell(x_{n_1+1}^{k-\ell-2\ell}))$. According to (5.20),

$$(E_{n_1+1,0} - E_{0,n+n_1+1})|_{\mathcal{B}} = x_{n_1+1} \partial_{x_0} - x_0 \partial_{y_{n_1+1}}. \quad (5.27)$$

Moreover, as operators on \mathcal{B} ,

$$\begin{aligned}
& [E_{n_1+1,0} - E_{0,n+n_1+1}, T_0] \\
&= [x_{n_1+1}\partial_{x_0} - x_0\partial_{y_{n_1+1}}, \sum_{i=0}^{\infty} \frac{(-2)^i x_0^{2i}}{(2i)!} \Delta^i] \\
&= [x_{n_1+1}, \sum_{i=1}^{\infty} \frac{(-2)^i x_0^{2i}}{(2i)!} \Delta^i] \partial_{x_0} + x_{n_1+1} \sum_{i=1}^{\infty} \frac{(-2)^i x_0^{2i-1}}{(2i-1)!} \Delta^i \\
&= -[\sum_{i=1}^{\infty} \frac{i(-2)^i x_0^{2i}}{(2i)!} \Delta^{i-1} \partial_{x_0} + \sum_{i=1}^{\infty} \frac{i(-2)^i x_0^{2i-1}}{(2i-1)!} \Delta^{i-1}] \partial_{y_{n_1+1}} - 2(T_1 \Delta) x_{n_1+1}, \tag{5.28}
\end{aligned}$$

$$\begin{aligned}
& [E_{n_1+1,0} - E_{0,n+n_1+1}, T_1] \\
&= [x_{n_1+1}\partial_{x_0} - x_0\partial_{y_{n_1+1}}, \sum_{i=0}^{\infty} \frac{(-2)^i x_0^{2i+1}}{(2i+1)!} \Delta^i] \\
&= [x_{n_1+1}, \sum_{i=1}^{\infty} \frac{(-2)^i x_0^{2i+1}}{(2i+1)!} \Delta^i] \partial_{x_0} + x_{n_1+1} \sum_{i=0}^{\infty} \frac{(-2)^i x_0^{2i}}{(2i)!} \Delta^i \\
&= -[\sum_{i=1}^{\infty} \frac{i(-2)^i x_0^{2i+1}}{(2i+1)!} \Delta^{i-1} \partial_{x_0} + \sum_{i=1}^{\infty} \frac{i(-2)^i x_0^{2i}}{(2i)!} \Delta^{i-1}] \partial_{y_{n_1+1}} + T_0 x_{n_1+1}. \tag{5.29}
\end{aligned}$$

If $T_0(\eta^\ell(x_{n_1+1}^{k-2\ell})) \in V$ for some $\ell \in \mathbb{N} + 1$, we have

$$\begin{aligned}
& (E_{n_1+1,0} - E_{0,n+n_1+1}) T_0(\eta^\ell(x_{n_1+1}^{k-2\ell})) \\
&= ([E_{n_1+1,0} - E_{0,n+n_1+1}, T_0] + T_0(E_{n_1+1,0} - E_{0,n+n_1+1}))(\eta^\ell(x_{n_1+1}^{k-2\ell})) \\
&= -[\sum_{i=1}^{\infty} \frac{i(-2)^i x_0^{2i-1}}{(2i-1)!} \Delta^{i-1} \partial_{y_{n_1+1}} + x_0 T_0 \partial_{y_{n_1+1}} + 2(T_1 \Delta) x_{n_1+1}](\eta^\ell(x_{n_1+1}^{k-2\ell})) \\
&= [T_1 \partial_{y_{n_1+1}} - 2(T_1 \Delta) x_{n_1+1}](\eta^\ell(x_{n_1+1}^{k-2\ell})) \\
&= \ell[1 - 2(m + n_1 - n_2 + \ell - 1)](\eta^{\ell-1}(x_{n_1+1}^{k-2(\ell-1)-1})) \\
&= \ell[3 - 2(m + n_1 - n_2 + \ell)] T_1(\eta^{\ell-1}(x_{n_1+1}^{k-2(\ell-1)-1})) \in V \tag{5.30}
\end{aligned}$$

by (4.63), (5.27) and (5.28). So $T_1(\eta^{\ell-1}(x_{n_1+1}^{k-2(\ell-1)-1})) \in V$. When $T_1(\eta^\ell(x_{n_1+1}^{k-2\ell-1})) \in V$ for some $\ell \in \mathbb{N}$, (5.27) and (5.29) yield

$$(E_{n_1+1,0} - E_{0,n+n_1+1}) T_1(\eta^\ell(x_{n_1+1}^{k-2\ell-1})) = T_0(\eta^\ell(x_{n_1+1}^{k-2\ell})) \in V. \tag{5.31}$$

By induction on ℓ , we have $x_{n_1+1}^k = T_0(x_{n_1+1}^k) \in V$.

Note

$$(E_{n+i,n+n_1+1} - E_{n_1+1,i})|_{\mathcal{B}} = y_i \partial_{y_{n_1+1}} + x_i x_{n_1+1} \quad \text{for } i \in \overline{1, n_1} \tag{5.32}$$

and

$$(E_{n_2+r,n_2} - E_{n+n_2,n+m_2+r})|_{\mathcal{A}} = x_{n_2+r} \partial_{x_{n_2}} + y_{n_2} y_{n_2+r} \quad \text{for } r \in \overline{1, n - n_2} \text{ if } n_2 < n \tag{5.33}$$

by (4.29) and (4.30). Repeatedly applying (5.32) and (5.33) to (5.31) with various $i \in \overline{1, n_1}$ and $r \in \overline{1, n - n_2}$ if $n_2 < n$, we have

$$[\prod_{i=1}^{n_1+1} x_i^{\alpha_i}] [\prod_{j=n_2}^n y_j^{\beta_j}] \in V \quad \text{for } \alpha_i, \beta_j \in \mathbb{N}; \alpha_{n_1+1} + \beta_{n_2} - \sum_{i=1}^{n_1} \alpha_i - \sum_{r=n_2+1}^n \beta_r = k. \tag{5.34}$$

Denote

$$I = \{0, \overline{n_1 + 1, n_2}, \overline{n + n_1 + 1, n + n_2}, \overline{2n + 1, 2n + 2m}\}. \quad (5.35)$$

Then the Lie subalgebra

$$\mathcal{G} = osp(2n + 1|2m) \bigcap \left(\sum_{i,j \in I} \mathbb{C} E_{i,j} \right) \cong osp(2(n_2 - n_1) + 1|2m). \quad (5.36)$$

Applying Theorem 5.1 to \mathcal{G} and $\mathbb{C}[x_0, x_{n_1+1}, \dots, x_{n_2}, y_{n_1+1}, \dots, y_{n_2}, \theta_1, \dots, \theta_m, \vartheta_1, \dots, \vartheta_m]$, we get

$$T_\iota(x^\alpha y^\beta \theta_{\vec{j}} \vartheta_{\vec{j}'}) \in V \quad (5.37)$$

for $\alpha, \beta \in \mathbb{N}^n$, $\vec{j} \in \Gamma_{k_1}$ and $\vec{j}' \in \Gamma_{k_2}$ such that $\beta_i = 0$ if $i \leq n_1$ and $\alpha_j = 0$ if $j > n_2$, and

$$\iota + k_1 + k_2 + \sum_{r=n_1+1}^{n_2} (\alpha_r + \beta_r) - \sum_{i=1}^{n_1} \alpha_i - \sum_{j=n_2+1}^n \beta_j = k. \quad (5.38)$$

Repeatedly applying (5.32) to (5.38) under above conditions with various $i \in \overline{1, n_1}$, we obtain (5.38) for $\alpha, \beta \in \mathbb{N}^n$, $\vec{j} \in \Gamma_{k_1}$ and $\vec{j}' \in \Gamma_{k_2}$ such that $\alpha_i = 0$ if $i > n_2$, and

$$\iota + k_1 + k_2 + \sum_{r=n_1+1}^{n_2} \alpha_r + \sum_{s=1}^{n_2} \beta_s - \sum_{i=1}^{n_1} \alpha_i - \sum_{j=n_2+1}^n \beta_j = k. \quad (5.39)$$

Observe

$$(E_{n_2+r, n_1+s} - E_{n+n_1+s, n+n_2+r})|_{\mathcal{B}} = y_{n_1+s} y_{n_2+r} + x_{n_2+r} \partial_{x_{n_1+s}} \quad (5.40)$$

for $r \in \overline{1, n - n_2}$ and $s \in \overline{1, n_2 - n_1}$ by (4.29) and (4.30). Repeatedly applying (5.40) to (5.38) with $\alpha_i = 0$ if $i > n_2$, we obtain $\mathcal{H}_{\langle k \rangle} \subset V$ by (5.25). So $\mathcal{H}_{\langle k \rangle}$ is an irreducible $osp(2n + 1|2m)$ -module.

Next we consider $\mathcal{H}_{\langle -k \rangle}$ with $k \in \mathbb{N} + 1$. Let U be any nonzero $osp(2n + 1|2m)$ -submodule of $\mathcal{H}'_{\langle -k \rangle}$. According to the arguments in paragraph of (4.68)-(4.73), U contains some $T_\iota(\eta^\ell(x_{n_1}^{k+\iota+2\ell}))$. Observe Note

$$(E_{n_1,0} - E_{0,n+n_1}) = \partial_{x_0} \partial_{x_{n_1}} - x_0 \partial_{y_{n_1}} \quad (5.41)$$

by (4.29) and (4.30). As operators on \mathcal{B} ,

$$\begin{aligned} & [E_{n_1,0} - E_{0,n+n_1}, T_0] \\ &= [\partial_{x_0} \partial_{x_{n_1}} - x_0 \partial_{y_{n_1}}, \sum_{i=0}^{\infty} \frac{(-2)^i x_0^{2i}}{(2i)!} \Delta^i] \\ &= \sum_{i=1}^{\infty} \frac{(-2)^i x_0^{2i-1}}{(2i-1)!} \Delta^i \partial_{x_{n_1}} - \partial_{x_0} \sum_{i=1}^{\infty} \frac{i(-2)^i x_0^{2i}}{(2i)!} \Delta^{i-1} \partial_{y_{n_1}} \\ &= -2T_1 \Delta \partial_{x_{n_1}} - \sum_{i=1}^{\infty} \frac{i(-2)^i x_0^{2i-1}}{(2i-1)!} \Delta^{i-1} \partial_{y_{n_1}} - \sum_{i=1}^{\infty} \frac{i(-2)^i x_0^{2i}}{(2i)!} \Delta^{i-1} \partial_{y_{n_1}} \partial_{x_0}, \end{aligned} \quad (5.42)$$

$$\begin{aligned} & [E_{n_1,0} - E_{0,n+n_1}, T_1] \\ &= [\partial_{x_0} \partial_{x_{n_1}} - x_0 \partial_{y_{n_1}}, \sum_{i=0}^{\infty} \frac{(-2)^i x_0^{2i+1}}{(2i+1)!} \Delta^i] \\ &= \sum_{i=0}^{\infty} \frac{(-2)^i x_0^{2i}}{(2i)!} \Delta^i \partial_{x_{n_1}} - \partial_{x_0} \sum_{i=1}^{\infty} \frac{i(-2)^i x_0^{2i+1}}{(2i+1)!} \Delta^{i-1} \partial_{y_{n_1}} \\ &= T_0 \partial_{x_{n_1}} - \sum_{i=1}^{\infty} \frac{i(-2)^i x_0^{2i}}{(2i)!} \Delta^{i-1} \partial_{y_{n_1}} - \sum_{i=1}^{\infty} \frac{i(-2)^i x_0^{2i+1}}{(2i+1)!} \Delta^{i-1} \partial_{y_{n_1}} \partial_{x_0}. \end{aligned} \quad (5.43)$$

If $T_0(\eta^\ell(x_{n_1}^{k+2\ell})) \in U$ for some $\ell \in \mathbb{N} + 1$, we have

$$\begin{aligned} & (E_{n_1,0} - E_{0,n+n_1})T_0(\eta^\ell(x_{n_1+1}^{k-2\ell})) \\ = & [T_1\partial_{y_{n_1}} - 2T_1\Delta\partial_{x_{n_1}}](\eta^\ell(x_{n_1+1}^{k+2\ell})) \\ = & (k+2\ell)\ell[1+2(n_1+k+\ell-n_2)]T_1(\eta^{\ell-1}(x_{n_1+1}^{k+2\ell-1})) \in V \end{aligned} \quad (5.44)$$

by (4.63), (5.41) and (5.42). So $T_1(\eta^{\ell-1}(x_{n_1}^{k+2(\ell-1)+1})) \in U$. When $T_1(\eta^\ell(x_{n_1}^{k+2\ell+1})) \in V$ for some $\ell \in \mathbb{N}$, (5.41) and (5.43) yield

$$(E_{n_1,0} - E_{0,n+n_1})T_1(\eta^\ell(x_{n_1}^{k+2\ell+1})) = (k+2\ell+1)T_0(\eta^\ell(x_{n_1}^{k+2\ell})) \in U. \quad (5.45)$$

By induction on ℓ , we have $x_{n_1}^k = T_0(x_{n_1}^k) \in U$.

According to (5.32) with $i = n_1$,

$$x_{n_1}^{k+k'}x_{n_1+1}^{k'} \in U \quad \text{for } k' \in \mathbb{N}. \quad (5.46)$$

Moreover,

$$(E_{i,n_1} - E_{n+n_1,n+i})|_{\mathcal{B}} = x_i\partial_{x_{n_1}} - y_{n_1}\partial_{y_i} \quad \text{for } i \in \overline{1, n_1-1} \quad (5.47)$$

by (4.29) and (4.30). Repeatedly applying (5.47) to (5.46) with various $i \in \overline{1, n_1-1}$, we have

$$\prod_{i=1}^{n_1+1} x_i^{\alpha_i} \in U \quad \text{for } \alpha_i \in \mathbb{N}; \alpha_{n_1+1} - \sum_{i=1}^{n_1} \alpha_i = -k. \quad (5.48)$$

Observe

$$(E_{n_2+r,n+1} - E_{1,n+n_2+r})|_{\mathcal{A}} = x_{n_2+r}\partial_{y_1} + y_{n_2+r}\partial_{x_1} \quad \text{for } r \in \overline{1, n-n_2} \text{ if } n_2 < n \quad (5.49)$$

by (4.31). Repeatedly applying (5.49) to (5.48) with various $r \in \overline{1, n-n_2}$ if $n_2 < n$, we find

$$[\prod_{i=1}^{n_1+1} x_i^{\alpha_i}][\prod_{j=n_2+1}^n y_j^{\beta_j}] \in U \quad \text{for } \alpha_i, \beta_j \in \mathbb{N}; \alpha_{n_1+1} - \sum_{i=1}^{n_1} \alpha_i - \sum_{j=n_2+1}^n \beta_j = -k. \quad (5.50)$$

By the same arguments from (5.36) to the end of the paragraph below (5.40) with k replaced by $-k$, we prove that $\mathcal{H}_{\langle -k \rangle}$ is an irreducible $osp(2n+1|2m)$ -module.

We calculate

$$[\Delta', \eta'] = 2 + 4(n_2 - n_1 - m) + 4[x_0\partial_{x_0} + \sum_{i=1}^n (x_i\partial_{x_i} + y_i\partial_{y_i}) + \sum_{r=1}^m (\theta_r\partial_{\theta_r} + \vartheta_r\partial_{\vartheta_r})]. \quad (5.51)$$

By the arguments in (3.52)-(3.58), $\mathcal{B} = \bigoplus_{\ell,k=0}^{\infty} (\eta')^\ell \mathcal{H}_{\langle k \rangle}$ is a direct sum of irreducible $osp(2n+1|2m)$ -submodules for any $k \in \mathbb{Z}$. \square

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